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FROM ONE BUBBLE TO SEVERAL BUBBLES: THE LOW-DIMENSIONAL CASE

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Abstract

We study in this paper sequences of solutions of elliptic PDE's with critical Sobolev growth on compact Riemannian manifolds. We prove some compactness results for such sequences which apply in particular to sequences of solutions of the Yamabe equation. We also underline the effect of the dimension and the geometry of the manifold on the blow-up behaviour of such sequences.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and $H_1^2(M)$ be the standard Sobolev space consisting of functions in $L^2(M)$ whose gradient is also in $L^2(M)$. We let h be a smooth function on M and consider equations like

$$\text{(E)} \qquad \Delta_g u + hu = u^{2^\star - 1}$$

where $\Delta_g = -\text{div}_g \nabla$ is the Laplace-Beltrami operator, and $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding of $H_1^2(M)$ into Lebesgue's spaces $L^q(M)$. Such equations have been the target of investigation for decades. They arise naturally in conformal geometry when $h = C(n)S_g$, $C(n) = (n-2)/4(n-1)$, where S_g is the scalar curvature of g. In this case, if u is a positive solution of (E), then the conformal metric $u^{\frac{4}{n-2}} g$ has constant scalar curvature. Equation (E) when $h = C(n)S_g$ is referred to as the Yamabe equation. Equations like (E) arise also naturally in the study of sharp Sobolev inequalities. Possible surveys on the Yamabe equation, including the final resolution of the Yamabe problem by Schoen [27], are [21, 28, 29]. Possible monographs on sharp Sobolev inequalities are [9] and [18].

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We let in this paper $(h_\varepsilon)_{\varepsilon>0}$ be a sequence of smooth functions on M verifying that there exists $h_0 \in C^2(M)$ such that $\Delta_g + h_0$ is coercive and such that

$$\lim_{\varepsilon\to 0} h_\varepsilon = h_0 \text{ in } C^2(M) \tag{0.1}$$

and we consider $(u_\varepsilon)_{\varepsilon>0}$ a sequence of smooth positive solutions of

$$\Delta_g u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^\star-1}\,. \tag{E$_\varepsilon$}$$

We assume that (u_ε) is of bounded energy in the sense that there exists $\Lambda > 0$ such that

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{2^\star}^{2^\star} \le \Lambda \tag{0.2}$$

where, as in the sequel, $\|\,.\,\|_p$ denotes the L^p-norm. Then, after passing to a subsequence,

$$\lim_{\varepsilon\to 0} u_\varepsilon = u_0 \text{ weakly in } H_1^2(M) \tag{0.3}$$

for some smooth nonnegative function u_0, solution of the limit equation

$$\Delta_g u_0 + h_0 u_0 = u_0^{2^\star-1}.$$

We assume in what follows that (0.2) and (0.3) hold. Since u_ε is positive, the maximum principle gives that either $u_0 \equiv 0$ or $u_0 > 0$. If (u_ε) is bounded in $L^\infty(M)$, then, thanks to standard elliptic theory,

$$\lim_{\varepsilon\to 0} u_\varepsilon = u_0 \text{ in } C^2(M)\,. \tag{0.4}$$

Throughout this paper, we assume that (0.4) is false so that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_\infty = +\infty. \tag{0.5}$$

Then the u_ε's develop a concentration phenomenon. This concentration phenomenon is well understood in $H_1^2(M)$, thanks for instance to Struwe [35]. Following Struwe [35], we get that, up to a subsequence,

$$u_\varepsilon = u_0 + \sum_{i=1}^{N} B_\varepsilon^i + R_\varepsilon \tag{0.6}$$

where $N \ge 1$ is an integer, the B_ε^i are bubbles obtained by rescaling fundamental positive solutions of the critical Euclidean equation $\Delta u =$

$u^{2^\star-1}$, and the R_ε's are lower order terms in $H_1^2(M)$, so that $R_\varepsilon \to 0$ in $H_1^2(M)$ as $\varepsilon \to 0$. A more precise definition of the bubbles is that

$$B_\varepsilon^i(x) = \left(\frac{\mu_{i,\varepsilon}}{\mu_{i,\varepsilon}^2 + a_n d_g(x_{i,\varepsilon}, x)^2}\right)^{\frac{n-2}{2}}$$

where $(x_{i,\varepsilon})$ is a converging sequence of points in M, and $(\mu_{i,\varepsilon})$ is a sequence of positive real numbers converging to 0 as $\varepsilon \to 0$. Here and in all the sequel,

$$a_n = \frac{1}{n\,(n-2)}.$$

Moreover, the bubbles do not interact at the H_1^2-level so that

$$\|u_\varepsilon\|_{2^\star}^{2^\star} = \|u_0\|_{2^\star}^{2^\star} + N\Lambda_{\min} + o\,(1)$$

where the minimum energy $\Lambda_{\min}$ is given by $\Lambda_{\min} = K_n^{-n/2}$, where K_n is the sharp constant for the Euclidean Sobolev inequality $\|u\|_{2^\star}^2 \leq K\|\nabla u\|_2^2$. We refer to the above description as the H_1^2-theory for blow-up. The C^0-theory, that we will use in this paper, was recently developed by Druet-Hebey-Robert [12, 13]. A special situation is when the u_ε's are of minimal energy, that is when $\Lambda = \Lambda_{\min}$ in (0.2). In such a case, thanks to the splitting of the energy in the above Struwe decomposition, we easily get that either $u_0 \equiv 0$ or $u_\varepsilon \to u_0$ in $C^2(M)$. In other words, either $u_0 \equiv 0$ or blow-up does not occur. A first and naïve question we address in this paper is whether or not such an alternative holds also when the bound on the energy in (0.2) is arbitrary, and, more generally, whether or not the dimension of the manifold has something to do with the vanishing or nonvanishing of u_0. An independent natural question when blow-up occurs is to determine the location of geometric concentration points. When the energy of the u_ε's is minimal and $n \geq 4$, see for instance Druet-Hebey [9] and Druet-Robert [14], we can prove that $h_0\,(\overline{x}) = C(n)S_g\,(\overline{x})$ where $\overline{x}$ is the geometric concentration point of the u_ε's. Another question we ask in this paper is whether or not this continues to hold when the bound on the energy in (0.2) is arbitrary. A positive answer to this question would provide another example of the criticality of the Yamabe equation. At last, we address the question of the compactness of solutions of (E_ε). This was first handled by Schoen [28] in the case of the Yamabe equation. We refer also to Schoen [29, 30].

We concentrate in this paper on the low-dimensional case, where $3 \leq n \leq 5$. We say that the u_ε's blow-up if (0.5) holds. We let then $\mathcal{S}$

be the set of the geometric concentration points of the u_ε's, defined as the set consisiting of the limits of the $x_{i,\varepsilon}$'s as $\varepsilon \to 0$. Independently, we say that compactness holds for the u_ε's if (0.4) holds. Our main result, which answers the above questions for low dimensions, is the following:

Theorem. *Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $3 \le n \le 5$. Let (u_ε) be a sequence of positive solutions of (E_ε). We assume that (0.1), (0.2), (0.3) hold. If the u_ε's blow-up, then $u_0 \equiv 0$, and when $n = 4, 5$, there exists $x_0 \in \mathcal{S}$ such that $h_0(x_0) = C(n)S_g(x_0)$, where $\mathcal{S}$ is the set of geometric concentration points, and $C(n)$ is as above. In particular, compactness holds for the u_ε's if $n = 4, 5$ and $h_0(x) \neq C(n)S_g(x)$ for all x in M. Compactness holds also for the u_ε's if $n = 3, 4, 5$ and $h_\varepsilon(x) \le C(n)S_g(x)$ for all x in M and all ε, with the additional condition that (M, g) has to be conformally distinct to the unit n-sphere if $h_0(x) = C(n)S_g(x)$ for all x in M.*

The compactness result of the theorem in its last part applies to the Yamabe equation, a situation where we recover the compactness result of Schoen [28]. In particular, if (M, g) is a compact Riemannian manifold of dimension $n = 3$, 4, 5, conformally distinct to the unit n-sphere, and if $(g_\varepsilon = u_\varepsilon^{4/(n-2)} g)$ is a sequence of conformal metrics to g of constant scalar curvature 1 and of bounded volume, then the sequence (u_ε) is precompact in $C^2(M)$. Our theorem, in its last part, was proved in dimension $n = 3$ by Li and Zhu [24]. The proof of our theorem relies on the C^0-theory for blow-up developed in Druet-Hebey-Robert [12]. The paper is organized as follows: in Section 1, we describe the C^0-theory developed in [12]. Section 2 is devoted to the estimate of the distance between concentration points and Section 3 deals with the special case of almost isolated concentration points. The analysis of the distance between concentration points was initiated (in the context of surfaces of constant mean curvature) by Brezis and Coron (see [4, 5, 6]). At last, in Section 4, we prove the theorem and give some results concerning higher dimensions. We also provide some instructive examples of blowing-up sequences of solutions of Equation (E_ε) in this last section.

1. A C^0-theory for blowing-up sequences of solutions of elliptic PDE's

In this section, we describe, and give some consequences of, the pointwise version of Struwe's result (see (0.6)) obtained in Druet-Hebey-

Robert [12]. We first recall the result which was proved in [12]. This result and one of its consequence (Claim 1 below) are the starting point for the analysis of Section 2.

Theorem ([12]). *Let (M,g) be a smooth compact Riemannian manifold of dimension $n\geq 3$, (h_ε) be a sequence of smooth functions on M satisfying (0.1), and (u_ε) be a sequence of smooth positive solutions of Equation (E_ε) satisfying (0.2) − (0.3). Assume that (0.5) holds, that is that $\|u_\varepsilon\|_\infty\to+\infty$ as $\varepsilon\to 0$. Then there exist $N\in\mathbb{N}^\star$, N converging sequences $(x_{i,\varepsilon})$ of points in M and N sequences $(\mu_{i,\varepsilon})$ of positive real numbers converging to 0 such that, after passing to a subsequence,*

$$\begin{aligned}&(1-\eta_\varepsilon)\,u_0\,(x)+\frac{1}{C}\sum_{i=1}^{N}\left(\frac{\mu_{i,\varepsilon}}{\mu_{i,\varepsilon}^2+a_n d_g\,(x_{i,\varepsilon},x)^2}\right)^{\frac{n-2}{2}}\\&\leq u_\varepsilon\,(x)\leq(1+\eta_\varepsilon)\,u_0\,(x)+C\sum_{i=1}^{N}\left(\frac{\mu_{i,\varepsilon}}{\mu_{i,\varepsilon}^2+d_g\,(x_{i,\varepsilon},x)^2}\right)^{\frac{n-2}{2}}\end{aligned}$$

for all $x\in M$ and all ε where $C>1$ is independent of ε and x and (η_ε), independent of x, is a sequence of positive real numbers converging to 0 as $\varepsilon\to 0$. In particular, the u_ε's are pointwisely controled on both sides by u_0 and standard bubbles.

This result has important applications when dealing with sharp Sobolev inequalities (see the monographs [9] and [18]). Other directions of research are the study of the energy function (see [19]). The above theorem is proved in Chapters 4 and 6 of Druet-Hebey-Robert [12]. Many asymptotic analysis of this kind are available in the minimal energy case: we cite among others the works of Atkinson-Peletier [2], Brezis-Peletier [7], Robert [25, 26] in the radially symmetric case on the Euclidean ball, the work of Han [15] when dealing with solutions u_ε of $\Delta u_\varepsilon=u_\varepsilon^{2^\star-1-\varepsilon}$ on arbitrary domains of $\mathbb{R}^n$, the work of Hebey-Vaugon [20] on arbitrary Riemannian manifolds with $h_\varepsilon\to+\infty$ as $\varepsilon\to 0$. In the case of the standard sphere, we refer also to Chang-Gursky-Yang [8], to Druet-Robert [14], to Li [22, 23] and to Schoen-Zhang [34]. One difficulty to get pointwise estimates when there are several bubbles is that bubbles do interact at a C^0-level except in dimension $n=3$ where one can prove a priori that the concentration points are isolated.

Let us come back to the above result. We let $(x_{i,\varepsilon})$ and $(\mu_{i,\varepsilon})$ be the points in M and the positive real numbers given by the theorem. We refer to Chapters 4 and 6 of [12] for all the following assertions: first, we have that

$$\text{(1.1)} \quad \text{for any } i, j \in \{1, \dots, N\}, \ i \neq j, \ \frac{d_g\left(x_{i,\varepsilon}, x_{j,\varepsilon}\right)^2}{\mu_{i,\varepsilon}\mu_{j,\varepsilon}} \to +\infty \ \text{ as } \varepsilon \to 0.$$

Then we have that for any $i \in \{1, \dots, N\}$,

$$\lim_{\varepsilon \to 0} \mu_{i,\varepsilon}^{\frac{n}{2}-1} u_\varepsilon \left(\exp_{x_{i,\varepsilon}}\left(\mu_{i,\varepsilon} x\right)\right) = u(x)$$

in $C^2_{\text{loc}}\left(\mathbb{R}^n \backslash \mathcal{S}_i\right)$ where

$$u(x) = \left(1 + a_n |x|^2\right)^{1-\frac{n}{2}}$$

is a solution of $\Delta_\xi u = u^{2^\star - 1}$ in $\mathbb{R}^n$, ξ the Euclidean metric, and

$$\mathcal{S}_i = \left\{ \lim_{\varepsilon \to 0} \frac{1}{\mu_{i,\varepsilon}} \exp^{-1}_{x_{i,\varepsilon}}\left(x_{j,\varepsilon}\right), \ j \neq i \text{ s.t. } x_{j,\varepsilon} \in B_{x_{i,\varepsilon}}\left(\frac{i_g(M)}{2}\right)\right\}.$$

In this definition, $i_g(M)$ is the injectivity radius of M and we assume that the limits exist, which is always the case after passing to a new subsequence.

Note that, as a direct consequence of the above theorem associated to standard elliptic theory, one gets that

$$\lim_{\varepsilon \to 0} u_\varepsilon = u_0 \ \text{ in } C^2_{\text{loc}}\left(M \backslash \mathcal{S}\right)$$

where

$$\text{(1.2)} \qquad \mathcal{S} = \left\{\overline{x}_i, \ i \in \{1, \dots, N\}\right\} \ \text{ with } \overline{x}_i = \lim_{\varepsilon \to 0} x_{i,\varepsilon}.$$

As proved in Section 6.3 of [12], the estimate of the theorem may be precised: let us define $S_0 \in C^0\left(M \times M\right)$ by

$$S_0(x, y) = \begin{cases} 1 & \text{if } x = y \\ \\ (n-2)\,\omega_{n-1} d_g(x, y)^{n-2}\, G_0(x, y) & \text{if } x \neq y \end{cases}$$

where G_0 is the Green function of $\Delta_g + h_0$, h_0 as in (0.1). The fact that $S_0 \in C^0\left(M \times M\right)$ comes from standard property of the Green function. We refer the reader to the appendix of [12] for estimates on Green's functions of linear elliptic operators on compact manifolds. We let (x_ε)

be a sequence of points in M such that $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. Then we have the following asymptotic estimate on $u_\varepsilon(x_\varepsilon)$ as $\varepsilon \to 0$:

$$u_\varepsilon(x_\varepsilon) = u_0(x_\varepsilon)\big(1+o(1)\big) + \sum_{i=1}^{N}\big(S_0(\overline{x}_i, x_0) + o(1)\big)\varphi_{i,\varepsilon}(x_\varepsilon) \tag{1.3}$$

where for $i = 1, \ldots, N$, $\varphi_{i,\varepsilon}$ is the standard bubble

$$\varphi_{i,\varepsilon}(x) = \left(\frac{\mu_{i,\varepsilon}}{\mu_{i,\varepsilon}^2 + a_n d_g(x_{i,\varepsilon}, x)^2}\right)^{\frac{n-2}{2}}. \tag{1.4}$$

Thanks to (1.1) and (1.3), we also have that Struwe's H_1^2-description holds with the bubbles of the theorem (see section 6.3 of [12]). Namely, we have that

$$u_\varepsilon = u_0 + \sum_{i=1}^{N}\varphi_{i,\varepsilon} + R_\varepsilon \tag{1.5}$$

with $\|R_\varepsilon\|_{H_1^2(M)} \to 0$ as $\varepsilon \to 0$. In the following, we shall always consider that the concentration points are ordered such that

$$\mu_{1,\varepsilon} \le \mu_{2,\varepsilon} \le \cdots \le \mu_{N,\varepsilon}. \tag{1.6}$$

As a last remark, note that standard elliptic theory leads thanks to (1.3) to the following if $u_0 \equiv 0$:

$$\lim_{\varepsilon\to 0} \mu_{N,\varepsilon}^{1-\frac{n}{2}} u_\varepsilon = a_n^{-\frac{n-2}{2}}(n-2)\,\omega_{n-1}\sum_{i=1}^{N}\left(\lim_{\varepsilon\to 0}\frac{\mu_{i,\varepsilon}}{\mu_{N,\varepsilon}}\right)^{\frac{n-2}{2}} G_0(\overline{x}_i, \,.\,) \tag{1.7}$$

in $C^2_{\mathrm{loc}}(M\backslash\mathcal{S})$, $\mathcal{S}$ and $\overline{x}_i$ as in (1.2).

We derive now from (1.3) an asymptotic estimate (Claim 1 below) we will often use in the sequel. We first set up some notations. We let $j \in \{1, \ldots, N\}$ and we let (δ_ε) be a sequence of positive real numbers converging to 0 as $\varepsilon \to 0$. We set

$$\mathcal{A}(j, \delta_\varepsilon) = \big\{i \in \{1, \ldots, N\} \text{ s.t. } d_g(x_{i,\varepsilon}, x_{j,\varepsilon}) = O(\delta_\varepsilon)\big\}. \tag{1.8}$$

Note that $j \in \mathcal{A}(j, \delta_\varepsilon)$. For $i \in \mathcal{A}(j, \delta_\varepsilon)$, we let

$$z_i = \lim_{\varepsilon\to 0}\frac{1}{\delta_\varepsilon}\exp^{-1}_{x_{j,\varepsilon}}(x_{i,\varepsilon}) \tag{1.9}$$

where the limits are assumed to exist (this is always true after passing to a subsequence). We set

$$\Sigma(j,\delta_\varepsilon)=\left\{z_i,\ i\in\mathcal{A}(j,\delta_\varepsilon)\right\}. \tag{1.10}$$

We let also

$$\lambda_\varepsilon(j,\delta_\varepsilon)=\left(\sup_{i\in\mathcal{A}(j,\delta_\varepsilon)}\mu_{i,\varepsilon}\right)^{1-\frac{n}{2}}\delta_\varepsilon^{n-2} \tag{1.11}$$

and

$$\begin{aligned}c(j,\delta_\varepsilon)=\sum_{k\notin\mathcal{A}(j,\delta_\varepsilon)}&\left(\lim_{\varepsilon\to 0}\lambda_\varepsilon(j,\delta_\varepsilon)\varphi_{k,\varepsilon}(x_{j,\varepsilon})\right)S_0(\overline{x}_k,\overline{x}_j)\\&+\lim_{\varepsilon\to 0}\big(\lambda_\varepsilon(j,\delta_\varepsilon)u_0(\overline{x}_j)\big)\end{aligned} \tag{1.12}$$

and for $k\in\mathcal{A}(j,\delta_\varepsilon)$,

$$\lambda_k(j,\delta_\varepsilon)=a_n^{-\frac{n-2}{2}}\lim_{\varepsilon\to 0}\left(\lambda_\varepsilon(j,\delta_\varepsilon)\delta_\varepsilon^{2-n}\mu_{k,\varepsilon}^{\frac{n}{2}-1}\right). \tag{1.13}$$

In (1.12), we assume that the limits do exist but they may be equal to $+\infty$. By convention, we say that $\lim_{\varepsilon\to 0}\big(\lambda_\varepsilon(j,\delta_\varepsilon)u_0(\overline{x}_j)\big)=0$ if $u_0\equiv 0$. We prove the following:

Claim 1. *Let $j\in\{1,\dots,N\}$ and let (δ_ε) be a sequence of positive real numbers converging to 0 as $\varepsilon\to 0$. We assume that the following holds:*

(H1) $\lambda_\varepsilon(j,\delta_\varepsilon)\delta_\varepsilon^{1-\frac{n}{2}}\to+\infty$ *as* $\varepsilon\to 0$.

(H2) $c(j,\delta_\varepsilon)<+\infty$.

Then, after passing to a subsequence, we have that

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon(j,\delta_\varepsilon)u_\varepsilon\left(\exp_{x_{j,\varepsilon}}(\delta_\varepsilon z)\right)=H(z)$$

in $C^2_{\mathrm{loc}}\big(\mathbb{R}^n\backslash\Sigma(j,\delta_\varepsilon)\big)$ *where*

$$H(z)=\sum_{k\in\mathcal{A}(j,\delta_\varepsilon)}\frac{\lambda_k(j,\delta_\varepsilon)}{|z-z_k|^{n-2}}+c(j,\delta_\varepsilon).$$

All the notations of this claim were introduced in (1.8)-(1.13).

Proof. Let $j \in \{1, \dots, N\}$ and let $(\delta_\varepsilon)_{\varepsilon>0}$ be a sequence of positive real numbers converging to 0 as $\varepsilon \to 0$. Assume that assumptions (H1) and (H2) of Claim 1 hold. We let $0 < \delta < \frac{i_g(M)}{2}$ and we set for $z \in B_0\left(\delta\delta_\varepsilon^{-1}\right)$, the Euclidean ball of center 0 and radius $\delta\delta_\varepsilon^{-1}$,

$$(1.14) \qquad \begin{aligned} &w_\varepsilon(z) = \delta_\varepsilon^{\frac{n}{2}-1} u_\varepsilon\left(\exp_{x_{j,\varepsilon}}(\delta_\varepsilon z)\right) \text{ and} \\ &g_\varepsilon(z) = \exp^\star_{x_{j,\varepsilon}} g(\delta_\varepsilon z). \end{aligned}$$

Since $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, we have that

$$(1.15) \qquad \lim_{\varepsilon\to 0} g_\varepsilon = \xi \text{ in } C^2_{\text{loc}}(\mathbb{R}^n).$$

Note also that g_ε is controled on both sides by ξ in the sense of bilinear forms. Since u_ε verifies Equation (E_ε), we have that w_ε verifies

$$(1.16) \qquad \Delta_{g_\varepsilon} w_\varepsilon + \delta_\varepsilon^2 h_\varepsilon\left(\exp_{x_{j,\varepsilon}}(\delta_\varepsilon z)\right) w_\varepsilon = w_\varepsilon^{2^\star-1}$$

in $B_0\left(\delta\delta_\varepsilon^{-1}\right)$. We claim that for any $R > 0$, there exists $C_R > 0$ independent of ε such that

$$(1.17) \qquad \lambda_\varepsilon(j,\delta_\varepsilon)\,\delta_\varepsilon^{1-\frac{n}{2}} \|w_\varepsilon\|_{L^\infty\left(B_0(R)\setminus\cup_{k\in\mathcal{A}(j,\delta_\varepsilon)} B_{z_k}\left(\frac{1}{R}\right)\right)} \le C_R$$

for all $\varepsilon > 0$ where $\mathcal{A}(j,\delta_\varepsilon)$ is as defined in (1.8), z_k is as in (1.9) and $\lambda_\varepsilon(j,\delta_\varepsilon)$ is as in (1.11). In order to prove (1.17), we let $R > 0$ and we let (z_ε) be a sequence of points in $B_0(R) \setminus \cup_{k\in\mathcal{A}(j,\delta_\varepsilon)} B_{z_k}\left(\frac{1}{R}\right)$. After passing to a subsequence, we may assume that $\lim_{\varepsilon\to 0} z_\varepsilon = z_0$. We let $x_\varepsilon = \exp_{x_{j,\varepsilon}}(\delta_\varepsilon z_\varepsilon)$ and we write thanks to (1.3) and (1.14) that

$$(1.18) \qquad \begin{aligned} \lambda_\varepsilon(j,\delta_\varepsilon)\,\delta_\varepsilon^{1-\frac{n}{2}} w_\varepsilon(z_\varepsilon) &= \lambda_\varepsilon(j,\delta_\varepsilon)\, u_0(x_\varepsilon)\left(1+o(1)\right) \\ &\quad + \lambda_\varepsilon(j,\delta_\varepsilon) \sum_{i=1}^N \left(S_0(\overline{x}_k,\overline{x}_j) + o(1)\right)\varphi_{k,\varepsilon}(x_\varepsilon). \end{aligned}$$

Thanks to (1.12) and to assumption (H2), we have that, up to a subsequence,

$$(1.19) \qquad \lim_{\varepsilon\to 0} \lambda_\varepsilon(j,\delta_\varepsilon)\, u_0(x_\varepsilon) = \lim_{\varepsilon\to 0}\left(\lambda_\varepsilon(j,\delta_\varepsilon)\, u_0(\overline{x}_j)\right)$$

where this limit is finite and is by convention equal to 0 if $u_0 \equiv 0$. Let $k \in \mathcal{A}(j,\delta_\varepsilon)$. By (1.11), we have that

$$\left(\frac{\mu_{k,\varepsilon}}{\delta_\varepsilon}\right)^{\frac{n}{2}-1} \le \delta_\varepsilon^{\frac{n}{2}-1}\lambda_\varepsilon(j,\delta_\varepsilon)^{-1}$$

which leads thanks to (H1) to $\mu_{k,\varepsilon} = o(\delta_\varepsilon)$. For $\varepsilon > 0$ small enough, we have thanks to (1.15) that $d_g(x_\varepsilon, x_{k,\varepsilon}) \ge \frac{1}{2R}\delta_\varepsilon$ so that $\frac{d_g(x_\varepsilon, x_{k,\varepsilon})}{\mu_{k,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$. This leads with (1.4) to

$$\begin{aligned}\lambda_\varepsilon(j,\delta_\varepsilon)\varphi_{k,\varepsilon}(x_\varepsilon) &= \left[a_n^{-\frac{n-2}{2}} + o(1)\right]\lambda_\varepsilon(j,\delta_\varepsilon)\mu_{k,\varepsilon}^{\frac{n}{2}-1} d_g(x_\varepsilon, x_{k,\varepsilon})^{2-n}\\ &= \lambda_k(j,\delta_\varepsilon)\left|z_0 - z_k\right|^{2-n} + o(1)\end{aligned}$$

where $\lambda_k(j,\delta_\varepsilon)$ is defined by (1.13) and is finite thanks to (1.11). Since $k \in \mathcal{A}(j,\delta_\varepsilon)$ and $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, we clearly have that $\overline{x}_k = \overline{x}_j$ so that $S_0(\overline{x}_k, \overline{x}_j) = 1$. Thus we have obtained that

(1.20)
$$\text{for any } k \in \mathcal{A}(j,\delta_\varepsilon)\ , \quad \lim_{\varepsilon\to 0}\lambda_\varepsilon(j,\delta_\varepsilon)\big(S_0(\overline{x}_k,\overline{x}_j) + o(1)\big)\varphi_{k,\varepsilon}(x_\varepsilon) = \lambda_k(j,\delta_\varepsilon)\left|z_0 - z_k\right|^{2-n}.$$

Let now $k \notin \mathcal{A}(j,\delta_\varepsilon)$. Since $d_g(x_{j,\varepsilon}, x_\varepsilon) = O(\delta_\varepsilon)$ and $\frac{d_g(x_{j,\varepsilon}, x_{k,\varepsilon})}{\delta_\varepsilon} \to +\infty$ as $\varepsilon \to 0$, we have that

$$\lim_{\varepsilon\to 0}\frac{d_g(x_\varepsilon, x_{k,\varepsilon})}{d_g(x_{j,\varepsilon}, x_{k,\varepsilon})} = 1$$

so that

$$\lim_{\varepsilon\to 0}\frac{\varphi_{k,\varepsilon}(x_\varepsilon)}{\varphi_{k,\varepsilon}(x_{j,\varepsilon})} = 1.$$

Thus we obtain that for any $k \notin \mathcal{A}(j,\delta_\varepsilon)$,

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon(j,\delta_\varepsilon)\varphi_{k,\varepsilon}(x_\varepsilon) = \lim_{\varepsilon\to 0}\lambda_\varepsilon(j,\delta_\varepsilon)\varphi_{k,\varepsilon}(x_{j,\varepsilon})$$

which does exist, after passing to a subsequence, and is finite thanks to (H2). Combining (1.18) with (1.19), (1.20) and this last relation, we get that

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon(j,\delta_\varepsilon)\delta_\varepsilon^{1-\frac{n}{2}} w_\varepsilon(z_\varepsilon) = H(z_0)$$

where H is as in Claim 1. In particular (1.17) is proved. Standard elliptic theory permits then to conclude thanks to (1.14) – (1.17) that Claim 1 holds. q.e.d.

2. Estimating the distance between bubbles

For any $j \in \{1, \ldots, N\}$, we set

$$\mathcal{A}_j = \left\{ i \neq j \ \text{ s.t. } \ \mu_{j,\varepsilon} = O\left(\mu_{i,\varepsilon}\right) \right\} \tag{2.1}$$

and

$$r_{j,\varepsilon}^2 = \begin{cases} \min\limits_{i \in \mathcal{A}_j} \left(\dfrac{\mu_{j,\varepsilon}}{\mu_{i,\varepsilon}} d_g\left(x_{i,\varepsilon}, x_{j,\varepsilon}\right)^2 + \mu_{i,\varepsilon}\mu_{j,\varepsilon} \right) & \text{if } u_0 \equiv 0 \\ \min\left\{ \min\limits_{i \in \mathcal{A}_j} \left(\dfrac{\mu_{j,\varepsilon}}{\mu_{i,\varepsilon}} d_g\left(x_{i,\varepsilon}, x_{j,\varepsilon}\right)^2 + \mu_{i,\varepsilon}\mu_{j,\varepsilon} \right); \mu_{j,\varepsilon} \right\} & \text{if } u_0 \not\equiv 0. \end{cases} \tag{2.2}$$

If $\mathcal{A}_j = \emptyset$ (which is possible only for $j = N$) and $u_0 \equiv 0$, we let $r_{j,\varepsilon} = 1$ for all $\varepsilon > 0$. Note that, thanks to (1.1),

$$\lim_{\varepsilon \to 0} \frac{r_{j,\varepsilon}}{\mu_{j,\varepsilon}} = +\infty \ \text{ for all } j \in \{1, \ldots, N\}\,. \tag{2.3}$$

The aim of this section is to get an estimate of $r_{j,\varepsilon}$ in terms of $\mu_{j,\varepsilon}$. If $r_{j,\varepsilon}$ does not converge to 0 as $\varepsilon \to 0$, we say that $(x_{j,\varepsilon}, \mu_{j,\varepsilon})$ is an almost isolated concentration point. We deal with almost isolated concentration points in Section 3. We treat in the following claim the case when $r_{j,\varepsilon} \to 0$ as $\varepsilon \to 0$:

Claim 2. *If $n = 3$, there exists $\delta_0 > 0$ such that $\liminf_{\varepsilon \to 0} r_{j,\varepsilon} \geq \delta_0$ for all $j \in \{1, \ldots, N\}$. In other words, the concentration points are isolated in dimension $n = 3$. Assume now that $n \geq 4$. Let $j \in \{1, \ldots, N\}$ be such that $r_{j,\varepsilon} \to 0$ as $\varepsilon \to 0$. Then the following assertions hold:*

(i) *After passing to a subsequence, we have that*

$$\lim_{\varepsilon \to 0} r_{j,\varepsilon}^{n-2} \mu_{j,\varepsilon}^{1-\frac{n}{2}} u_\varepsilon \left(\exp_{x_{j,\varepsilon}}\left(r_{j,\varepsilon} z\right) \right) = a_n^{-\frac{n-2}{2}} \left(\frac{1}{|z|^{n-2}} + h_j\left(z\right) \right)$$

in $C^2_{\text{loc}}\left(\mathbb{R}^n \backslash \Sigma_j\right)$ where

$$\begin{aligned} \Sigma_j &= \left\{ z_{j,k},\, k \in \mathcal{B}_j \right\} \cup \{0\}\ , \\ \mathcal{B}_j &= \left\{ k \neq j,\, d_g\left(x_{j,\varepsilon}, x_{k,\varepsilon}\right) = O\left(r_{j,\varepsilon}\right) \right\} \text{ and} \\ z_{j,k} &= \lim_{\varepsilon \to 0} \frac{1}{r_{j,\varepsilon}} \exp^{-1}_{x_{j,\varepsilon}}\left(x_{k,\varepsilon}\right),\ k \in \mathcal{B}_j \end{aligned}$$

and where

$$h_j(z) = \sum_{k \in \mathcal{A}_j \cap \mathcal{B}_j} \frac{\lambda_{j,k}}{|z - z_{j,k}|^{n-2}} + c_j$$

with

$$\lambda_{j,k} = \left(\lim_{\varepsilon \to 0} \frac{\mu_{k,\varepsilon}}{\mu_{j,\varepsilon}} \right)^{\frac{n}{2}-1}$$

and

$$\begin{aligned} a_n^{-\frac{n-2}{2}} c_j \quad &= \sum_{k \in \mathcal{A}_j \setminus \mathcal{B}_j} S_0(\overline{x}_k, \overline{x}_j) \lim_{\varepsilon \to 0} \left(r_{j,\varepsilon}^{n-2} \mu_{j,\varepsilon}^{1-\frac{n}{2}} \varphi_{k,\varepsilon}(x_{j,\varepsilon}) \right) \\ &\quad + \lim_{\varepsilon \to 0} \left(r_{j,\varepsilon}^{n-2} \mu_{j,\varepsilon}^{1-\frac{n}{2}} u_0(\overline{x}_j) \right) . \end{aligned}$$

(ii) *We have that*

$$\left(h_0(\overline{x}_j) - C(4) S_g(\overline{x}_j) + o(1) \right) r_{j,\varepsilon}^2 \ln \left(\frac{1}{\mu_{j,\varepsilon}} \right) = 2 h_j(0)$$

when $n = 4$ *and that*

$$\begin{aligned} &\left(h_0(\overline{x}_j) - C(n) S_g(\overline{x}_j) + o(1) \right) r_{j,\varepsilon}^{n-2} \mu_{j,\varepsilon}^{4-n} \\ &= a_n^{2-n} K_n^{\frac{n}{2}} \frac{(n-2)^2 (n-4)}{8(n-1)} \omega_{n-1} h_j(0) \end{aligned}$$

when $n \geq 5$*. Here,* $C(n) = \frac{n-2}{4(n-1)}$ *and* $a_n = \frac{1}{n(n-2)}$*. Moreover,*

$$h_j(0) > 0.$$

(iii) *If* $h_0(\overline{x}_j) > C(n) S_g(\overline{x}_j)$*, we have that* $\nabla h_j(0) = 0$*.*

Proof. Let $j \in \{1, \ldots, N\}$ be such that

$$\lim_{\varepsilon \to 0} r_{j,\varepsilon} = 0. \tag{2.4}$$

We want to apply Claim 1 to $(j, r_{j,\varepsilon})$. We let $\mathcal{B}_j$ be as in Claim 2 so that $\mathcal{B}_j \cup \{j\} = \mathcal{A}(j, r_{j,\varepsilon})$, $\mathcal{A}(j, r_{j,\varepsilon})$ as in (1.8). We verify that the assumptions of Claim 1 are satisfied by $(j, r_{j,\varepsilon})$. First, by (2.4), $r_{j,\varepsilon} \to 0$ as $\varepsilon \to 0$. Let $k \in \mathcal{A}_j \cap \mathcal{B}_j$. Then we have that $d_g(x_{j,\varepsilon}, x_{k,\varepsilon}) = O(r_{j,\varepsilon})$ so that, using (1.1), (2.1) and (2.2), we obtain that $\mu_{k,\varepsilon} = O(\mu_{j,\varepsilon})$. Thus

$$\text{for any } k \in \mathcal{A}_j \cap \mathcal{B}_j,\ \mu_{k,\varepsilon} = O(\mu_{j,\varepsilon}). \tag{2.5}$$

If $k \notin \mathcal{A}_j$, by the definition (2.1) of $\mathcal{A}_j$, we have that $\mu_{k,\varepsilon} = o(\mu_{j,\varepsilon})$. Thus for any $k \in \mathcal{B}_j$, $\mu_{k,\varepsilon} = O(\mu_{j,\varepsilon})$. This gives that there exists $C > 0$ such that

$$C \leq \lambda_\varepsilon(j, r_{j,\varepsilon}) \, \mu_{j,\varepsilon}^{\frac{n}{2}-1} r_{j,\varepsilon}^{2-n} \leq 1 \tag{2.6}$$

for all $\varepsilon > 0$ where $\lambda_\varepsilon(j, r_{j,\varepsilon})$ is defined by (1.11). By (2.3), we thus get that

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon(j, r_{j,\varepsilon}) \, r_{j,\varepsilon}^{1-\frac{n}{2}} = +\infty$$

so that assumption (H1) of Claim 1 is satisfied. Let $k \notin \mathcal{B}_j$, $k \neq j$, that is $k \in \{1, \dots, N\}$ such that

$$\frac{d_g(x_{k,\varepsilon}, x_{j,\varepsilon})}{r_{j,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0.$$

We write thanks to (1.4) and (2.6) that

$$\left(\lambda_\varepsilon(j, r_{j,\varepsilon}) \, \varphi_{k,\varepsilon}(x_{j,\varepsilon})\right)^{\frac{2}{n-2}} \leq \frac{r_{j,\varepsilon}^2}{\mu_{j,\varepsilon}} \frac{\mu_{k,\varepsilon}}{\mu_{k,\varepsilon}^2 + a_n d_g(x_{j,\varepsilon}, x_{k,\varepsilon})^2}.$$

If $k \in \mathcal{A}_j$, we get then by (2.2) that $\lambda_\varepsilon(j, r_{j,\varepsilon}) \, \varphi_{k,\varepsilon}(x_{j,\varepsilon}) = O(1)$. If $k \notin \mathcal{A}_j$, then $\mu_{k,\varepsilon} = o(\mu_{j,\varepsilon})$ and we write that

$$\left(\lambda_\varepsilon(j, r_{j,\varepsilon}) \, \varphi_{k,\varepsilon}(x_{j,\varepsilon})\right)^{\frac{2}{n-2}} \leq \frac{1}{a_n} \frac{\mu_{k,\varepsilon}}{\mu_{j,\varepsilon}} \frac{r_{j,\varepsilon}^2}{d_g(x_{k,\varepsilon}, x_{j,\varepsilon})^2} = o(1)$$

since $k \notin \mathcal{B}_j$, $k \neq j$. If $u_0 \not\equiv 0$, then $r_{j,\varepsilon}^2 \leq \mu_{j,\varepsilon}$ by (2.2) so that (2.6) gives that $\lambda_\varepsilon(j, r_{j,\varepsilon}) \leq 1$. We have thus proved thanks to (1.12) that assumption (H2) of Claim 1 holds. Applying Claim 1 to $(j, r_{j,\varepsilon})$, we get that assertion (i) of Claim 2 holds for j thanks to (2.5) and (2.6). In order to compute c_j, note that, as just proved, if $k \notin \mathcal{A}_j$, $k \notin \mathcal{B}_j$, $k \neq j$, then $r_{j,\varepsilon}^{n-2} \mu_{j,\varepsilon}^{1-\frac{n}{2}} \varphi_{k,\varepsilon}(x_{j,\varepsilon}) \to 0$ as $\varepsilon \to 0$. We claim now that

$$h_j(0) > 0. \tag{2.7}$$

Let us prove this claim. If $\mathcal{A}_j = \emptyset$, then $r_{j,\varepsilon} = 1$ if $u_0 \equiv 0$, a situation which is excluded by (2.4). If $u_0 \not\equiv 0$ and $r_{j,\varepsilon}^2 = \mu_{j,\varepsilon}$, we have that

$$c_j \geq a_n^{\frac{n-2}{2}} u_0(\overline{x}_j) > 0$$

so that $h_j(0) > 0$. Assume now that $\mathcal{A}_j \neq \emptyset$ and that there exists $k \in \mathcal{A}_j$ such that

$$r_{j,\varepsilon}^2 = \frac{\mu_{j,\varepsilon}}{\mu_{k,\varepsilon}} d_g(x_{j,\varepsilon}, x_{k,\varepsilon})^2 + \mu_{j,\varepsilon}\mu_{k,\varepsilon}.$$

If $k \in \mathcal{B}_j$, then $h_j(0) > 0$ since, by (2.1), $\lambda_{j,i} > 0$ for all $i \in \mathcal{A}_j \cap \mathcal{B}_j$. If $k \notin \mathcal{B}_j$, we write thanks to (1.4) that

$$\left(r_{j,\varepsilon}^{n-2}\mu_{j,\varepsilon}^{1-\frac{n}{2}}\varphi_{k,\varepsilon}(x_{j,\varepsilon})\right)^{\frac{2}{n-2}} = \frac{\mu_{k,\varepsilon}^2 + d_g(x_{j,\varepsilon}, x_{k,\varepsilon})^2}{\mu_{k,\varepsilon}^2 + a_n d_g(x_{j,\varepsilon}, x_{k,\varepsilon})^2} \geq 1$$

so that $h_j(0) > 0$ in this last case. Relation (2.7) is proved. Note that (2.7) is the second part of assertion ii) for $i = j$.

Let us set

$$\mathcal{C}_j = \left\{k \in \mathcal{B}_j \text{ s.t. } z_{j,k} = 0\right\} \tag{2.8}$$

where $z_{j,k}$ is as in Claim 2. Let $k \in \mathcal{C}_j$. If $k \notin \mathcal{A}_j$, then $\mu_{k,\varepsilon} = o(\mu_{j,\varepsilon})$ and if $k \in \mathcal{A}_j$, it follows from (1.1), (2.2) and (2.8) that $\mu_{k,\varepsilon} = o(\mu_{j,\varepsilon})$ also. Thus

$$\mu_{k,\varepsilon} = o(\mu_{j,\varepsilon}) \text{ for all } k \in \mathcal{C}_j. \tag{2.9}$$

We let

$$s_{j,k,\varepsilon}^2 = \frac{\mu_{k,\varepsilon}}{\mu_{j,\varepsilon}} d_g(x_{k,\varepsilon}, x_{j,\varepsilon})^2 + \mu_{j,\varepsilon}\mu_{k,\varepsilon} \tag{2.10}$$

for $k \in \mathcal{C}_j$. Note that, by (2.3), (2.8) and (2.9), we have that

$$s_{j,k,\varepsilon} = o(r_{j,\varepsilon}) \text{ for all } k \in \mathcal{C}_j. \tag{2.11}$$

We let now $\mathcal{D}_j$ be a subset of $\mathcal{C}_j$ and $(R_k)_{k\in\mathcal{D}_j}$ be a sequence of positive real numbers such that

$$\text{for any } k, k' \in \mathcal{D}_j,\ k \neq k',\ \frac{d_g\left(x_{k,\varepsilon}, x_{k',\varepsilon}\right)}{s_{j,k,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0 \tag{2.12}$$

and such that

$$\text{for any } k' \in \mathcal{C}_j,\ \exists \text{ a unique } k \in \mathcal{D}_j \text{ such that} \tag{2.13}$$
$$\limsup_{\varepsilon\to 0} \frac{d_g\left(x_{k,\varepsilon}, x_{k',\varepsilon}\right)}{s_{j,k,\varepsilon}} \leq \frac{R_k}{10} \text{ and } \limsup_{\varepsilon\to 0} \frac{s_{j,k',\varepsilon}}{s_{j,k,\varepsilon}} \leq \frac{R_k}{10}.$$

We claim that there exists $C > 0$ independent of ε such that for any $k \in \mathcal{D}_j$,

$$\text{for any } x \in B_{x_{k,\varepsilon}}\left(R_k s_{j,k,\varepsilon}\right) \backslash B_{x_{k,\varepsilon}}\left(\frac{R_k}{4} s_{j,k,\varepsilon}\right), \quad |\nabla u_\varepsilon|_g(x) \le C\mu_{k,\varepsilon}^{\frac{n}{2}-1} s_{j,k,\varepsilon}^{1-n}, \;\; u_\varepsilon(x) \le C\mu_{k,\varepsilon}^{\frac{n}{2}-1} s_{j,k,\varepsilon}^{2-n}\,. \tag{2.14}$$

The proof of this claim is based on Claim 1. We check that we can apply Claim 1 to $(k, s_{j,k,\varepsilon})$ for $k \in \mathcal{D}_j$. First, by (2.4) and (2.11), it is clear that $s_{j,k,\varepsilon} \to 0$ as $\varepsilon \to 0$. Let $i \in \mathcal{A}(k, s_{j,k,\varepsilon})$, that is $i \in \{1, \ldots, N\}$ such that $d_g(x_{i,\varepsilon}, x_{k,\varepsilon}) = O(s_{j,k,\varepsilon})$. Since $d_g(x_{i,\varepsilon}, x_{j,\varepsilon}) \le d_g(x_{i,\varepsilon}, x_{k,\varepsilon}) + d_g(x_{k,\varepsilon}, x_{j,\varepsilon})$, we get thanks to (2.8) and (2.11) that $i \in \mathcal{C}_j$. By (2.13), we then have that $s_{j,i,\varepsilon} = O(s_{j,k,\varepsilon})$. By (1.1), (2.9) and (2.10), $d_g(x_{i,\varepsilon}, x_{k,\varepsilon}) = O(s_{j,k,\varepsilon}) = o\big(d_g(x_{j,\varepsilon}, x_{k,\varepsilon})\big)$. This leads thanks to the definition (2.10) of $s_{j,i,\varepsilon}$ and $s_{j,k,\varepsilon}$ to $\mu_{i,\varepsilon} = O(\mu_{k,\varepsilon})$. Thus $\mu_{i,\varepsilon} = O(\mu_{k,\varepsilon})$ for all $i \in \mathcal{A}(k, s_{j,k,\varepsilon})$. This gives the existence of some $C > 0$ such that

$$C \le \lambda_\varepsilon(k, s_{j,k,\varepsilon})\, \mu_{k,\varepsilon}^{\frac{n}{2}-1} s_{j,k,\varepsilon}^{2-n} \le 1 \tag{2.15}$$

where $\lambda_\varepsilon(k, s_{j,k,\varepsilon})$ is defined by (1.11). This implies in particular that

$$\lambda_\varepsilon(k, s_{j,k,\varepsilon})\, s_{j,k,\varepsilon}^{1-\frac{n}{2}} \ge C\left(\frac{s_{j,k,\varepsilon}^2}{\mu_{k,\varepsilon}^2}\right)^{\frac{n-2}{4}} \ge C\left(\frac{d_g(x_{j,\varepsilon}, x_{k,\varepsilon})^2}{\mu_{k,\varepsilon}\mu_{j,\varepsilon}}\right)^{\frac{n-2}{4}}$$

so that $\lambda_\varepsilon(k, s_{j,k,\varepsilon})\, s_{j,k,\varepsilon}^{1-\frac{n}{2}} \to +\infty$ as $\varepsilon \to 0$ thanks to (1.1). This proves that assumption (H1) of Claim 1 holds for $(k, s_{j,k,\varepsilon})$. Let $i \notin \mathcal{A}(k, s_{j,k,\varepsilon})$, that is $i \in \{1, \ldots, N\}$ such that

$$\frac{d_g(x_{i,\varepsilon}, x_{k,\varepsilon})}{s_{j,k,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0.$$

In order to estimate $\big(\lambda_\varepsilon(k, s_{j,k,\varepsilon})\, \varphi_{i,\varepsilon}(x_{k,\varepsilon})\big)$, we write thanks to (1.4) and (2.15) that

$$\big(\lambda_\varepsilon(k, s_{j,k,\varepsilon})\, \varphi_{i,\varepsilon}(x_{k,\varepsilon})\big)^{\frac{2}{n-2}} \le \frac{s_{j,k,\varepsilon}^2}{\mu_{k,\varepsilon}} \frac{\mu_{i,\varepsilon}}{\mu_{i,\varepsilon}^2 + a_n d_g(x_{i,\varepsilon}, x_{k,\varepsilon})^2}. \tag{2.16}$$

We distinguish several cases. First, assume that $i \in \mathcal{C}_j$. Then, by (2.12), (2.13) and thanks to the fact that $\frac{d_g(x_{i,\varepsilon}, x_{k,\varepsilon})}{s_{j,k,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$, we get that

$$\frac{d_g\left(x_{k,\varepsilon},x_{i,\varepsilon}\right)}{s_{j,i,\varepsilon}}\to+\infty \text{ as } \varepsilon\to 0. \tag{2.17}$$

This gives that

$$\begin{aligned}\frac{d_g\left(x_{i,\varepsilon},x_{k,\varepsilon}\right)^2}{\mu_{i,\varepsilon}^2} &= \frac{d_g\left(x_{k,\varepsilon},x_{i,\varepsilon}\right)^2}{s_{j,i,\varepsilon}^2}\left(\frac{d_g\left(x_{i,\varepsilon},x_{j,\varepsilon}\right)^2}{\mu_{i,\varepsilon}\mu_{j,\varepsilon}}+\frac{\mu_{j,\varepsilon}}{\mu_{i,\varepsilon}}\right)\\ &\to +\infty \text{ as } \varepsilon\to 0\end{aligned}$$

thanks to (1.1). Thus (2.16) becomes in this case

$$\left(\lambda_\varepsilon\left(k,s_{j,k,\varepsilon}\right)\varphi_{i,\varepsilon}\left(x_{k,\varepsilon}\right)\right)^{\frac{2}{n-2}}=O\left(\frac{\mu_{i,\varepsilon}}{\mu_{k,\varepsilon}}\frac{s_{j,k,\varepsilon}^2}{d_g\left(x_{i,\varepsilon},x_{k,\varepsilon}\right)^2}\right).$$

We write now thanks to (2.10) that

$$s_{j,k,\varepsilon}^2\le 2\frac{\mu_{k,\varepsilon}}{\mu_{j,\varepsilon}}\left(d_g\left(x_{i,\varepsilon},x_{k,\varepsilon}\right)^2+d_g\left(x_{i,\varepsilon},x_{j,\varepsilon}\right)^2\right)+\mu_{j,\varepsilon}\mu_{k,\varepsilon}$$

so that we get that

$$\begin{aligned}&\left(\lambda_\varepsilon\left(k,s_{j,k,\varepsilon}\right)\varphi_{i,\varepsilon}\left(x_{k,\varepsilon}\right)\right)^{\frac{2}{n-2}}\\ &=O\left(\frac{\mu_{i,\varepsilon}}{\mu_{j,\varepsilon}}\right)+O\left(\frac{\mu_{i,\varepsilon}}{\mu_{j,\varepsilon}}\frac{d_g\left(x_{i,\varepsilon},x_{j,\varepsilon}\right)^2}{d_g\left(x_{i,\varepsilon},x_{k,\varepsilon}\right)^2}+\frac{\mu_{i,\varepsilon}\mu_{j,\varepsilon}}{d_g\left(x_{i,\varepsilon},x_{k,\varepsilon}\right)^2}\right)\\ &=O\left(\frac{\mu_{i,\varepsilon}}{\mu_{j,\varepsilon}}\right)+O\left(\frac{s_{j,i,\varepsilon}^2}{d_g\left(x_{i,\varepsilon},x_{k,\varepsilon}\right)^2}\right)=o(1)\end{aligned}$$

since $i\in\mathcal{C}_j$ and thanks to (2.9) and (2.17). Second, assume that $i=j$. In this case, (2.16) becomes

$$\left(\lambda_\varepsilon\left(k,s_{j,k,\varepsilon}\right)\varphi_{i,\varepsilon}\left(x_{k,\varepsilon}\right)\right)^{\frac{2}{n-2}}\le\frac{\mu_{j,\varepsilon}^2+d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)^2}{\mu_{j,\varepsilon}^2+a_n d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)^2}\le n(n-2).$$

Third, assume that $i\notin\mathcal{C}_j$ and that $i\in\mathcal{A}_j$. Then we write with (2.16) that

$$\begin{aligned}&\left(\lambda_\varepsilon\left(k,s_{j,k,\varepsilon}\right)\varphi_{i,\varepsilon}\left(x_{k,\varepsilon}\right)\right)^{\frac{2}{n-2}}\\ &\le\left(\frac{d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)^2}{\mu_{j,\varepsilon}}+\mu_{j,\varepsilon}\right)\frac{\mu_{i,\varepsilon}}{\mu_{i,\varepsilon}^2+a_n d_g\left(x_{i,\varepsilon},x_{k,\varepsilon}\right)^2}.\end{aligned}$$

Since $i \notin \mathcal{C}_j$ and $i \neq j$, we have by (2.8) that

$$d_g(x_{i,\varepsilon}, x_{k,\varepsilon}) = \big(1 + o(1)\big) d_g(x_{i,\varepsilon}, x_{j,\varepsilon}).$$

Since $k \in \mathcal{C}_j$, we also have that $d_g(x_{j,\varepsilon}, x_{k,\varepsilon}) = o(r_{j,\varepsilon})$. Since $i \in \mathcal{A}_j$, this leads with (2.1) and (2.2) to

$$\big(\lambda_\varepsilon(k, s_{j,k,\varepsilon})\varphi_{i,\varepsilon}(x_{k,\varepsilon})\big)^{\frac{2}{n-2}} = O\left(\frac{\mu_{i,\varepsilon}^2 + d_g(x_{j,\varepsilon}, x_{i,\varepsilon})^2}{\mu_{i,\varepsilon}^2 + \frac{d_g(x_{i,\varepsilon}, x_{j,\varepsilon})^2}{n(n-2)}}\right) = O(1).$$

At last, assume that $i \notin \mathcal{C}_j$, $i \notin \mathcal{A}_j$ and $i \neq j$. Since $i \notin \mathcal{C}_j$ and $i \neq j$, we have as above that

$$d_g(x_{i,\varepsilon}, x_{k,\varepsilon}) = \big(1 + o(1)\big) d_g(x_{i,\varepsilon}, x_{j,\varepsilon}).$$

Since $i \notin \mathcal{A}_j$, $i \neq j$, we know that $\mu_{i,\varepsilon} = o(\mu_{j,\varepsilon})$ so that, by (1.1), $\frac{d_g(x_{i,\varepsilon}, x_{j,\varepsilon})}{\mu_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$. Thus (2.16) becomes in this case

$$\begin{aligned} &\big(\lambda_\varepsilon(k, s_{j,k,\varepsilon})\varphi_{i,\varepsilon}(x_{k,\varepsilon})\big)^{\frac{2}{n-2}} \\ &= O\left(\frac{\mu_{i,\varepsilon}}{\mu_{j,\varepsilon}} \frac{d_g(x_{j,\varepsilon}, x_{k,\varepsilon})^2}{d_g(x_{i,\varepsilon}, x_{j,\varepsilon})^2}\right) + O\left(\frac{\mu_{i,\varepsilon}\mu_{j,\varepsilon}}{d_g(x_{i,\varepsilon}, x_{j,\varepsilon})^2}\right) \\ &= o(1) \end{aligned}$$

since $\mu_{i,\varepsilon} = o(\mu_{j,\varepsilon})$ and $i \notin \mathcal{C}_j$, $i \neq j$, and thanks to (1.1). Thus we have proved that

$$\text{for any } i \notin \mathcal{A}(k, s_{j,k,\varepsilon}),\ \ \lambda_\varepsilon(k, s_{j,k,\varepsilon})\varphi_{i,\varepsilon}(x_{k,\varepsilon}) = O(1).$$

Assume that $u_0 \not\equiv 0$. Then we can write thanks to (2.10) and (2.15) that

$$\begin{aligned} \lambda_\varepsilon(k, s_{j,k,\varepsilon})^{\frac{2}{n-2}} &\leq \frac{s_{j,k,\varepsilon}^2}{\mu_{k,\varepsilon}} \leq \frac{d_g(x_{j,\varepsilon}, x_{k,\varepsilon})^2}{\mu_{j,\varepsilon}} + \mu_{j,\varepsilon} \\ &= o\left(\frac{r_{j,\varepsilon}^2}{\mu_{j,\varepsilon}}\right) + o(1) = o(1) \end{aligned}$$

using (2.2) and (2.8). Thus assumption (H2) of Claim 1 is verified by $(k, s_{j,k,\varepsilon})$. We can apply Claim 1. This proves (2.14) thanks to the choice of R_k we made. Indeed, by (2.11), any $i \in \mathcal{A}(k, s_{j,k,\varepsilon})$ belongs to $\mathcal{C}_j$ and by (2.13), we get then that

$$\Sigma(k, s_{j,k,\varepsilon}) \subset B_0\left(\frac{R_k}{5}\right).$$

This clearly ends the proof of (2.14) thanks to (2.15).

We let now $\eta : [0, +\infty[\mapsto \mathbb{R}$ be a smooth function verifying that $0 \le \eta \le 1$, $\eta \equiv 1$ on $\left[0, \frac{1}{4}\right]$ and $\eta \equiv 0$ on $\left[\frac{1}{2}, +\infty\right[$. We set

$$\sigma_{j,\varepsilon} = \prod_{k \in \mathcal{D}_j} \left(1 - \eta\left(\frac{d_g(x_{k,\varepsilon}, .)}{R_k s_{j,k,\varepsilon}}\right)\right) \text{ and } v_{j,\varepsilon} = \sigma_{j,\varepsilon} u_\varepsilon. \tag{2.18}$$

We claim that there exists $C > 0$ independent of ε such that

$$v_{j,\varepsilon} \le C\varphi_{j,\varepsilon} \text{ in } B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon}) \tag{2.19}$$

where $\delta_j \in \mathbb{R}_+^\star$ is fixed such that

$$\delta_j \le \frac{1}{2} \min\left\{|z_{j,k}|,\, k \in \mathcal{B}_j \backslash \mathcal{C}_j,\, k \ne j\right\}. \tag{2.20}$$

It is possible to find such a δ_j thanks to (2.8). The proof of (2.19) is based on (1.3) which gives that there exists $C > 0$ independent of ε such that

$$\frac{u_\varepsilon(x)}{\varphi_{j,\varepsilon}(x)} \le C\left(1 + \frac{\|u_0\|_\infty}{\varphi_{j,\varepsilon}(x)} + \sum_{k \ne j} \frac{\varphi_{k,\varepsilon}(x)}{\varphi_{j,\varepsilon}(x)}\right)$$

for all $x \in B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})$. As well as $r_{j,\varepsilon}$ and δ_j have been chosen so that

$$\frac{\|u_0\|_\infty}{\varphi_{j,\varepsilon}(x)} + \sum_{k \notin \mathcal{C}_j} \frac{\varphi_{k,\varepsilon}(x)}{\varphi_{j,\varepsilon}(x)} \le C$$

for all $x \in B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})$ for some $C > 0$ independent of ε (see the proof that the assumptions of Claim 1 hold for $(j, r_{j,\varepsilon})$), it is easily checked that $s_{j,k,\varepsilon}$ has been defined so that

$$\frac{\varphi_{k,\varepsilon}(x)}{\varphi_{j,\varepsilon}(x)} \le C$$

for all $x \in M \backslash B_{x_{k,\varepsilon}}\left(\frac{R_k}{4} s_{j,k,\varepsilon}\right)$. These two assertions, whose proofs are left to the reader, imply (2.19).

We set

$$\Lambda_{j,\varepsilon} = \Big\{ (y,\nu,\theta) \in M \times \mathbb{R}_+^\star \times \mathbb{R} \text{ s.t. } d_g(x_{j,\varepsilon}, y) \le \mu_{j,\varepsilon}\,,\ \frac{1}{2} \le \frac{\nu}{\mu_{j,\varepsilon}} \le 2\,,\ -\frac{1}{2} \le \theta \le \frac{1}{2} \Big\} \tag{2.21}$$

and we let $(y_{j,\varepsilon}, \nu_{j,\varepsilon}, \theta_{j,\varepsilon}) \in \Lambda_{j,\varepsilon}$ be such that

$$J_{j,\varepsilon}(y_{j,\varepsilon}, \nu_{j,\varepsilon}, \theta_{j,\varepsilon}) = \min_{(y,\nu,\theta)\in \Lambda_{j,\varepsilon}} J_{j,\varepsilon}(y,\nu,\theta) \tag{2.22}$$

where

$$J_{j,\varepsilon}(y,\nu,\theta) = \int_M \left| \nabla \left(\eta \left(\frac{d_g(y,.)}{2\delta_j r_{j,\varepsilon}} \right) \left(v_{j,\varepsilon} - (1+\theta)\,\psi_{y,\nu} \right) \right) \right|_g^2 dv_g$$

with

$$\psi_{y,\nu}(x) = \left(\frac{\nu}{\nu^2 + a_n d_g(x,y)^2} \right)^{\frac{n-2}{2}}. \tag{2.23}$$

We claim first that

$$J_{j,\varepsilon}(y_{j,\varepsilon}, \nu_{j,\varepsilon}, \theta_{j,\varepsilon}) \to 0 \text{ as } \varepsilon \to 0. \tag{2.24}$$

In order to prove (2.24), we first note that $(x_{j,\varepsilon}, \mu_{j,\varepsilon}, 0) \in \Lambda_{j,\varepsilon}$ so that

$$J_{j,\varepsilon}(y_{j,\varepsilon}, \nu_{j,\varepsilon}, \theta_{j,\varepsilon}) \le J_{j,\varepsilon}(x_{j,\varepsilon}, \mu_{j,\varepsilon}, 0)\,.$$

We write now that

$$\begin{aligned} J_{j,\varepsilon}(x_{j,\varepsilon}, \mu_{j,\varepsilon}, 0) &= \int_M \left| \nabla \left(\eta \left(\frac{d_g(x_{j,\varepsilon},.)}{2\delta_j r_{j,\varepsilon}} \right) (v_{j,\varepsilon} - \varphi_{j,\varepsilon}) \right) \right|_g^2 dv_g \\ &\le \frac{C}{r_{j,\varepsilon}^2} \int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon}) \setminus B_{x_{j,\varepsilon}}\left(\frac{\delta_j}{2} r_{j,\varepsilon}\right)} (v_{j,\varepsilon} - \varphi_{j,\varepsilon})^2\, dv_g \\ &\quad + C \int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})} \left| \nabla (v_{j,\varepsilon} - \varphi_{j,\varepsilon}) \right|_g^2 dv_g \end{aligned}$$

where $C > 0$ is some constant independent of ε. Thanks to (2.3) and (2.19), it is easily checked that

$$\lim_{\varepsilon\to 0} \frac{1}{r_{j,\varepsilon}^2} \int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon}) \setminus B_{x_{j,\varepsilon}}\left(\frac{\delta_j}{2} r_{j,\varepsilon}\right)} (v_{j,\varepsilon} - \varphi_{j,\varepsilon})^2\, dv_g = 0$$

so that we obtain that

$$J_{j,\varepsilon}\left(y_{j,\varepsilon},\nu_{j,\varepsilon},\theta_{j,\varepsilon}\right)=O\left(\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)\right|_g^2\,dv_g\right). \tag{2.25}$$

We write with (1.5) that

$$\begin{aligned}
&\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)\right|_g^2\,dv_g\\
&\le C\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\Big(\left(\sigma_{j,\varepsilon}-1\right)\varphi_{j,\varepsilon}\Big)\right|_g^2\,dv_g\\
&\quad+C\sum_{i\neq j}\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\left(\sigma_{j,\varepsilon}\varphi_{i,\varepsilon}\right)\right|_g^2\,dv_g\\
&\quad+C\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\left(\sigma_{j,\varepsilon}u_0\right)\right|_g^2\,dv_g+C\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\left(\sigma_{j,\varepsilon}R_\varepsilon\right)\right|_g^2\,dv_g.
\end{aligned} \tag{2.26}$$

with $\|R_\varepsilon\|_{H_1^2(M)}\to 0$ as $\varepsilon\to 0$. We write that

$$\begin{aligned}
&\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\left(\sigma_{j,\varepsilon}R_\varepsilon\right)\right|_g^2\,dv_g\\
&=O\left(\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\sigma_{j,\varepsilon}\right|_g^2 R_\varepsilon^2\,dv_g\right)\\
&\quad+O\left(\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla R_\varepsilon\right|_g^2\sigma_{j,\varepsilon}^2\,dv_g\right)\\
&=O\left(\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\sigma_{j,\varepsilon}\right|_g^2 R_\varepsilon^2\,dv_g\right)+o(1)
\end{aligned}$$

since $R_\varepsilon\to 0$ in $H_1^2(M)$ as $\varepsilon\to 0$. Thanks to (2.12) and (2.18), we have that

$$\begin{aligned}
&\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})}\left|\nabla\sigma_{j,\varepsilon}\right|_g^2 R_\varepsilon^2\,dv_g\\
&=O\left(\sum_{k\in\mathcal{D}_j}\frac{1}{s_{j,k,\varepsilon}^2}\int_{B_{x_{k,\varepsilon}}\left(\frac{1}{2}R_k s_{j,k,\varepsilon}\right)}R_\varepsilon^2\,dv_g\right)\\
&=O\left(\sum_{k\in\mathcal{D}_j}\frac{1}{s_{j,k,\varepsilon}^2}\mathrm{Vol}_g\left(B_{x_{k,\varepsilon}}\left(\frac{1}{2}R_k s_{j,k,\varepsilon}\right)\right)^{\frac{2}{n}}\|R_\varepsilon\|_{2^\star}^2\right)
\end{aligned}$$

with Hölder's inequalities. By Sobolev's inequalities, since $R_\varepsilon \to 0$ in $H_1^2(M)$ as $\varepsilon \to 0$, we finally obtain that

$$\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})} \left|\nabla\left(\sigma_{j,\varepsilon} R_\varepsilon\right)\right|_g^2 \, dv_g = o(1)\,. \tag{2.27}$$

In the same way, it is easily checked thanks to (2.12), (2.18) and to the fact that we assumed (2.4) that

$$\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})} \left|\nabla\left(\sigma_{j,\varepsilon} u_0\right)\right|_g^2 \, dv_g = o(1)\,. \tag{2.28}$$

Let now $i \in \{1, \dots, N\}$, $i \neq j$. We write thanks to (2.12) and (2.18) that

$$\begin{aligned}
&\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})} \left|\nabla\left(\sigma_{j,\varepsilon}\varphi_{i,\varepsilon}\right)\right|_g^2 \, dv_g \\
&= O\left(\sum_{k\in\mathcal{D}_j} \frac{1}{s_{j,k,\varepsilon}^2} \int_{B_{x_{k,\varepsilon}}\left(\frac{1}{2}R_k s_{j,k,\varepsilon}\right)\setminus B_{x_{k,\varepsilon}}\left(\frac{1}{4}R_k s_{j,k,\varepsilon}\right)} \varphi_{i,\varepsilon}^2 \, dv_g\right) \\
&\quad + O\left(\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})\setminus \cup_{k\in\mathcal{D}_j} B_{x_{k,\varepsilon}}\left(\frac{1}{4}R_k s_{j,k,\varepsilon}\right)} \left|\nabla\varphi_{i,\varepsilon}\right|_g^2 \, dv_g\right) \\
&= O\left(\left(\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})\setminus \cup_{k\in\mathcal{D}_j} B_{x_{k,\varepsilon}}\left(\frac{1}{4}R_k s_{j,k,\varepsilon}\right)} \varphi_{i,\varepsilon}^{2^\star} \, dv_g\right)^{\frac{2}{2^\star}}\right) \\
&\quad + O\left(\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})\setminus \cup_{k\in\mathcal{D}_j} B_{x_{k,\varepsilon}}\left(\frac{1}{4}R_k s_{j,k,\varepsilon}\right)} \left|\nabla\varphi_{i,\varepsilon}\right|_g^2 \, dv_g\right).
\end{aligned}$$

If $i \in \mathcal{C}_j$, (1.1) and (2.10) give that $\frac{s_{j,i,\varepsilon}}{\mu_{i,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$ so that, by (2.12) and (2.13), we have that

$$B_{x_{i,\varepsilon}}\left(R\mu_{i,\varepsilon}\right) \subset \bigcup_{k\in\mathcal{D}_j} B_{x_{k,\varepsilon}}\left(\frac{1}{4}R_k s_{j,k,\varepsilon}\right)$$

for all $R > 0$ as soon as ε is small enough. Direct computations give then that

$$\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})\setminus \cup_{k\in\mathcal{D}_j} B_{x_{k,\varepsilon}}\left(\frac{1}{4}R_k s_{j,k,\varepsilon}\right)} \left(\varphi_{i,\varepsilon}^{2^\star} + \left|\nabla\varphi_{i,\varepsilon}\right|_g^2\right) dv_g = o(1)\,.$$

If $i \notin \mathcal{C}_j$, direct computations, distinguishing whether $i \in \mathcal{A}_j$ or not, give the same result thanks to (1.1), (2.1), (2.2) and (2.20). Thus we have that for any $i \in \{1, \dots, N\}$, $i \neq j$,

$$\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})} \left|\nabla\left(\sigma_{j,\varepsilon}\varphi_{i,\varepsilon}\right)\right|_g^2 \, dv_g = o(1). \tag{2.29}$$

At last, we write thanks to (2.12) and (2.18) that

$$\begin{aligned}
&\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})} \left|\nabla\left(\left(\sigma_{j,\varepsilon}-1\right)\varphi_{j,\varepsilon}\right)\right|_g^2 \, dv_g \\
&= O\left(\sum_{k\in\mathcal{D}_j} \frac{1}{s_{j,k,\varepsilon}^2} \int_{B_{x_{k,\varepsilon}}\left(\frac{R_k}{2}s_{j,k,\varepsilon}\right)} \varphi_{j,\varepsilon}^2 \, dv_g\right) \\
&\quad + O\left(\sum_{k\in\mathcal{D}_j} \int_{B_{x_{k,\varepsilon}}\left(\frac{R_k}{2}s_{j,k,\varepsilon}\right)} \left|\nabla\varphi_{j,\varepsilon}\right|_g^2 \, dv_g\right).
\end{aligned}$$

By (1.1), (2.9) and (2.10), we have that

$$\frac{d_g\left(x_{k,\varepsilon}, x_{j,\varepsilon}\right)}{s_{j,k,\varepsilon}} \to +\infty \text{ as } \varepsilon \to 0$$

so that

$$\varphi_{j,\varepsilon}^2 = O\left(\mu_{j,\varepsilon}^{2-n}\left(1+\frac{d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)^2}{\mu_{j,\varepsilon}^2}\right)^{2-n}\right) \text{ and}$$

$$\left|\nabla\varphi_{j,\varepsilon}\right|_g^2 = O\left(\mu_{j,\varepsilon}^{-n}\frac{d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)^2}{\mu_{j,\varepsilon}^2}\left(1+\frac{d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)^2}{\mu_{j,\varepsilon}^2}\right)^{-n}\right)$$

in $B_{x_{k,\varepsilon}}\left(\frac{R_k}{2}s_{j,k,\varepsilon}\right)$ for all $k \in \mathcal{D}_j$. Thus we have that

$$\begin{aligned}
&\int_{B_{x_{j,\varepsilon}}(\delta_j r_{j,\varepsilon})} \left|\nabla\left(\left(\sigma_{j,\varepsilon}-1\right)\varphi_{j,\varepsilon}\right)\right|_g^2 \, dv_g \\
&= O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{s_{j,k,\varepsilon}}{\mu_{j,\varepsilon}}\right)^{n-2}\left(1+\frac{d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)^2}{\mu_{j,\varepsilon}^2}\right)^{2-n}\right) \\
&\quad + O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{s_{j,k,\varepsilon}}{\mu_{j,\varepsilon}}\right)^{n}\frac{d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)^2}{\mu_{j,\varepsilon}^2}\left(1+\frac{d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)^2}{\mu_{j,\varepsilon}^2}\right)^{-n}\right).
\end{aligned}$$

By (2.9) and (2.10), we have that for any $k \in \mathcal{D}_j$,

$$\frac{s_{j,k,\varepsilon}}{\mu_{j,\varepsilon}} = o\left(\frac{d_g\left(x_{j,\varepsilon}, x_{k,\varepsilon}\right)}{\mu_{j,\varepsilon}}\right) + o\left(1\right).$$

This easily leads to

$$\int_{B_{x_{j,\varepsilon}}\left(\delta_j r_{j,\varepsilon}\right)} \left|\nabla\big(\left(\sigma_{j,\varepsilon}-1\right)\varphi_{j,\varepsilon}\big)\right|_g^2 \, dv_g = o\left(1\right).$$

Coming back to (2.26) with (2.27) – (2.29) and this last relation, we obtain that

$$\lim_{\varepsilon\to 0}\int_{B_{x_{j,\varepsilon}}\left(\delta_j r_{j,\varepsilon}\right)} \left|\nabla\left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)\right|_g^2 \, dv_g = 0. \tag{2.30}$$

Thanks to (2.25), this proves (2.24). Let us prove now that

$$\lim_{\varepsilon\to 0}\theta_{j,\varepsilon} = 0\ ,\ \ \lim_{\varepsilon\to 0}\frac{\mu_{j,\varepsilon}}{\nu_{j,\varepsilon}} = 1 \ \text{ and } \ \lim_{\varepsilon\to 0}\frac{d_g\left(x_{j,\varepsilon}, y_{j,\varepsilon}\right)}{\mu_{j,\varepsilon}} = 0. \tag{2.31}$$

Note that, by the definition (2.21) of $\Lambda_{j,\varepsilon}$, all these limits exist after passing to a subsequence. We write that

$$J_{j,\varepsilon}\left(y_{j,\varepsilon},\nu_{j,\varepsilon},\theta_{j,\varepsilon}\right) = \int_M \left|\nabla\Big(\eta_{j,\varepsilon}\big(v_{j,\varepsilon}-\left(1+\theta_{j,\varepsilon}\right)\psi_{j,\varepsilon}\big)\Big)\right|_g^2 \, dv_g$$

where

$$\eta_{j,\varepsilon} = \eta\left(\frac{d_g\left(y_{j,\varepsilon},\,.\,\right)}{2\delta_j r_{j,\varepsilon}}\right) \ \text{ and } \ \psi_{j,\varepsilon} = \psi_{y_{j,\varepsilon},\nu_{j,\varepsilon}}.$$

Then we write that

$$\begin{aligned} J_{j,\varepsilon}\left(y_{j,\varepsilon},\nu_{j,\varepsilon},\theta_{j,\varepsilon}\right) = {} & \left\|\nabla\left(\eta_{j,\varepsilon}v_{j,\varepsilon}\right)\right\|_2^2 + \left(1+\theta_{j,\varepsilon}\right)^2\left\|\nabla\left(\eta_{j,\varepsilon}\psi_{j,\varepsilon}\right)\right\|_2^2 \\ & - 2\left(1+\theta_{j,\varepsilon}\right)\int_M \big(\nabla\left(\eta_{j,\varepsilon}v_{j,\varepsilon}\right), \nabla\left(\eta_{j,\varepsilon}\psi_{j,\varepsilon}\right)\big)_g \, dv_g. \end{aligned} \tag{2.32}$$

This leads in particular to

$$J_{j,\varepsilon}\left(y_{j,\varepsilon},\nu_{j,\varepsilon},\theta_{j,\varepsilon}\right) \geq \big(\left\|\nabla\left(\eta_{j,\varepsilon}v_{j,\varepsilon}\right)\right\|_2 - \left(1+\theta_{j,\varepsilon}\right)\left\|\nabla\left(\eta_{j,\varepsilon}\psi_{j,\varepsilon}\right)\right\|_2\big)^2.$$

Thanks to (2.3), (2.19) and (2.30), it is easily checked by direct computations that $\lim_{\varepsilon\to 0}\left\|\nabla\left(\eta_{j,\varepsilon}v_{j,\varepsilon}\right)\right\|_2 = K_n^{-\frac{n}{4}}$. Independently, direct computations give thanks to (2.3) and to the definition (2.21) of $\Lambda_{j,\varepsilon}$

that $\|\nabla(\eta_{j,\varepsilon}\psi_{j,\varepsilon})\|_2 \to K_n^{-\frac{n}{4}}$ as $\varepsilon \to 0$. By (2.24), the above relation then gives that $\theta_{j,\varepsilon} \to 0$ as $\varepsilon \to 0$. Coming back to (2.32), we then get that

$$\lim_{\varepsilon\to 0}\int_M \left(\nabla(\eta_{j,\varepsilon}v_{j,\varepsilon}), \nabla(\eta_{j,\varepsilon}\psi_{j,\varepsilon})\right)_g dv_g = K_n^{-\frac{n}{2}}$$

which leads in turn thanks to (2.19) and (2.30) to

$$\lim_{\varepsilon\to 0}\int_M \left(\nabla(\eta_{j,\varepsilon}\varphi_{j,\varepsilon}), \nabla(\eta_{j,\varepsilon}\psi_{j,\varepsilon})\right)_g dv_g = K_n^{-\frac{n}{2}}.$$

But this last relation is easily seen to be possible if and only if $\frac{\mu_{j,\varepsilon}}{\nu_{j,\varepsilon}} \to 1$ and $\frac{d_g(x_{j,\varepsilon},y_{j,\varepsilon})}{\mu_{j,\varepsilon}} \to 0$ as $\varepsilon \to 0$. This ends the proof of (2.31).

Let $0 < \delta < \frac{i_g(M)}{2}$. We set for $x \in B_0\left(\delta r_{j,\varepsilon}^{-1}\right)$, the Euclidean ball of center 0 and radius $\delta r_{j,\varepsilon}^{-1}$,

$$\begin{aligned}
g_{j,\varepsilon}(x) &= \exp_{y_{j,\varepsilon}}^{\star} g(r_{j,\varepsilon}x), \\
u_{j,\varepsilon}(x) &= r_{j,\varepsilon}^{\frac{n}{2}-1} u_\varepsilon\left(\exp_{y_{j,\varepsilon}}(r_{j,\varepsilon}x)\right), \\
h_{j,\varepsilon}(x) &= h_\varepsilon\left(\exp_{y_{j,\varepsilon}}(r_{j,\varepsilon}x)\right), \\
\widetilde{\sigma}_{j,\varepsilon}(x) &= \sigma_{j,\varepsilon}\left(\exp_{y_{j,\varepsilon}}(r_{j,\varepsilon}x)\right) \text{ and} \\
\widetilde{v}_{j,\varepsilon}(x) &= \widetilde{\sigma}_{j,\varepsilon}(x)\, u_{j,\varepsilon}(x) = r_{j,\varepsilon}^{\frac{n}{2}-1} v_{j,\varepsilon}\left(\exp_{y_{j,\varepsilon}}(r_{j,\varepsilon}x)\right).
\end{aligned} \tag{2.33}$$

By (2.4), we know that

$$\lim_{\varepsilon\to 0} g_{j,\varepsilon} = \xi \text{ in } C^4_{\mathrm{loc}}(\mathbb{R}^n). \tag{2.34}$$

Note also that $g_{j,\varepsilon}$ is controled on both sides by the Euclidean metric in the sense of bilinear forms. Since u_ε verifies Equation (E_ε), $u_{j,\varepsilon}$ verifies

$$\Delta_{g_{j,\varepsilon}} u_{j,\varepsilon} + r_{j,\varepsilon}^2 h_{j,\varepsilon} u_{j,\varepsilon} = u_{j,\varepsilon}^{2^\star-1} \tag{2.35}$$

in $B_0\left(\delta r_{j,\varepsilon}^{-1}\right)$. Independently, using (2.3), (2.31) and (2.33), one gets since assertion (i) of Claim 2 holds for $i = j$, as proved above, that

$$\lim_{\varepsilon\to 0}\left(\frac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}\right)^{\frac{n}{2}-1} u_{j,\varepsilon} = a_n^{-\frac{n-2}{2}}\left(\frac{1}{|z|^{n-2}} + h_j(z)\right) \tag{2.36}$$

in $C^2_{\text{loc}}\left(\mathbb{R}^n\backslash\Sigma_j\right)$ where Σ_j and h_j are as in Claim 2. By (2.20), we have that

$$\Sigma_j \cap B_0\left(\delta_j\right) = \{0\}\,.$$

At last, (2.19) together with (2.31) gives the existence of some $C > 0$ such that for any $\varepsilon > 0$,

$$\widetilde{v}_{j,\varepsilon}\left(x\right) \le C\widetilde{\psi}_{j,\varepsilon} \text{ in } B_0\left(\delta_j\right) \tag{2.37}$$

where

$$\widetilde{\psi}_{j,\varepsilon}\left(x\right) = \left(\frac{\nu_{j,\varepsilon} r_{j,\varepsilon}}{\nu_{j,\varepsilon}^2 + a_n r_{j,\varepsilon}^2 |x|^2}\right)^{\frac{n-2}{2}}.$$

We write

$$\eta_j \widetilde{v}_{j,\varepsilon} = \left(1+\theta_{j,\varepsilon}\right)\eta_j\widetilde{\psi}_{j,\varepsilon} + w_{j,\varepsilon} \tag{2.38}$$

where $w_{j,\varepsilon} \in C_c^\infty\left(B_0\left(\delta_j\right)\right)$ and

$$\eta_j = \eta\left(\frac{\cdot}{2\delta_j}\right).$$

We express (2.22). Differentiating $J_{j,\varepsilon}$ with respect to θ, we obtain that

$$\int_{B_0(\delta_j)} \left(\nabla\left(\eta_j\widetilde{\psi}_{j,\varepsilon}\right), \nabla w_{j,\varepsilon}\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} = 0. \tag{2.39}$$

Differentiating $J_{j,\varepsilon}$ with respect to y, we get thanks to (2.36) and (2.38) that

$$\int_{B_0(\delta_j)} \left(\nabla\left(\eta_j\frac{\partial\widetilde{\psi}_{j,\varepsilon}}{\partial x_i}\right), \nabla w_{j,\varepsilon}\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} = O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) \tag{2.40}$$

for all $i = 1, \dots, n$. At last, differentiating $J_{j,\varepsilon}$ with respect to ν, we obtain thanks to (2.39) that

$$\int_{B_0(\delta_j)} \left(\nabla\left(\eta_j |x|^2 \left(1 + a_n\frac{r_{j,\varepsilon}^2|x|^2}{\nu_{j,\varepsilon}^2}\right)^{-\frac{n}{2}}\right), \nabla w_{j,\varepsilon}\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} = 0. \tag{2.41}$$

The aim is now to estimate $\int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|^2_{g_{j,\varepsilon}}\, dv_{g_{j,\varepsilon}}$. We write first thanks to (2.38) and (2.39) that

$$\begin{aligned}\int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|^2_{g_{j,\varepsilon}}\, dv_{g_{j,\varepsilon}} &= \int_{B_0(\delta_j)} \left(\nabla w_{j,\varepsilon}, \nabla\left(\eta_j\widetilde{v}_{j,\varepsilon}\right)\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} \\ &= \int_{B_0(\delta_j)} w_{j,\varepsilon}\Delta_{g_{j,\varepsilon}}\left(\eta_j\widetilde{v}_{j,\varepsilon}\right) dv_{g_{j,\varepsilon}}.\end{aligned}$$

Writing thanks to (2.33), (2.35), (2.36) and (2.38) that

$$\begin{aligned} w_{j,\varepsilon}\Delta_{g_{j,\varepsilon}}\left(\eta_j\widetilde{v}_{j,\varepsilon}\right) = &\left(\eta_j\widetilde{v}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon} - r_{j,\varepsilon}^2 h_{j,\varepsilon}\left(\eta_j\widetilde{v}_{j,\varepsilon}\right) w_{j,\varepsilon} \\ &+ O\left(\left|w_{j,\varepsilon}\right| \left|\nabla\widetilde{\sigma}_{j,\varepsilon}\right|_{g_{j,\varepsilon}} \left|\nabla u_{j,\varepsilon}\right|_{g_{j,\varepsilon}}\right) \\ &+ O\left(\left|w_{j,\varepsilon}\right| \left|\Delta_{g_{j,\varepsilon}}\widetilde{\sigma}_{j,\varepsilon}\right| u_{j,\varepsilon}\right) + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) \\ &+ O\left(\left|\widetilde{\sigma}_{j,\varepsilon} - \widetilde{\sigma}_{j,\varepsilon}^{2^\star-1}\right| u_{j,\varepsilon}^{2^\star-1}\left|w_{j,\varepsilon}\right|\right) \end{aligned}$$

in $B_0\left(\delta_j\right)$, we get that

(2.42)

$$\begin{aligned} &\int_{B_0(\delta_j)} \left|\nabla w_{j,\varepsilon}\right|_{g_{j,\varepsilon}}^2 dv_{g_{j,\varepsilon}} \\ &= \int_{B_0(\delta_j)} \left(\eta_j\widetilde{v}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} - r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} h_{j,\varepsilon}\left(\eta_j\widetilde{v}_{j,\varepsilon}\right) w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} \\ &\quad + O\left(\int_{B_0(\delta_j)} \left|w_{j,\varepsilon}\right| \left|\nabla\widetilde{\sigma}_{j,\varepsilon}\right|_{g_{j,\varepsilon}} \left|\nabla u_{j,\varepsilon}\right|_{g_{j,\varepsilon}}\, dv_{g_{j,\varepsilon}}\right) \\ &\quad + O\left(\int_{B_0(\delta_j)} \left|w_{j,\varepsilon}\right| \left|\Delta_{g_{j,\varepsilon}}\widetilde{\sigma}_{j,\varepsilon}\right| u_{j,\varepsilon}\, dv_{g_{j,\varepsilon}}\right) + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) \\ &\quad + O\left(\int_{B_0(\delta_j)} \left|\widetilde{\sigma}_{j,\varepsilon} - \widetilde{\sigma}_{j,\varepsilon}^{2^\star-1}\right| u_{j,\varepsilon}^{2^\star-1}\left|w_{j,\varepsilon}\right|\, dv_{g_{j,\varepsilon}}\right). \end{aligned}$$

For any $k \in \mathcal{D}_j$, we set

$$B_{j,k,\varepsilon} = \frac{1}{r_{j,\varepsilon}} \exp_{y_{j,\varepsilon}}^{-1}\left(B_{x_{k,\varepsilon}}\left(\frac{R_k}{2} s_{j,k,\varepsilon}\right) \backslash B_{x_{k,\varepsilon}}\left(\frac{R_k}{4} s_{j,k,\varepsilon}\right)\right).$$

Using (2.12), (2.14), (2.18), (2.33), Hölder's and Sobolev's inequalities, we get that

$$\begin{aligned}
&\int_{B_0(\delta_j)} |w_{j,\varepsilon}| \left|\nabla\widetilde{\sigma}_{j,\varepsilon}\right|_{g_{j,\varepsilon}} \left|\nabla u_{j,\varepsilon}\right|_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}} \\
&= O\left(\sum_{k\in\mathcal{D}_j} \frac{r_{j,\varepsilon}}{s_{j,k,\varepsilon}} \int_{B_{j,k,\varepsilon}} |w_{j,\varepsilon}| \left|\nabla u_{j,\varepsilon}\right|_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}}\right) \\
&= O\left(\sum_{k\in\mathcal{D}_j} \frac{r_{j,\varepsilon}}{s_{j,k,\varepsilon}} \frac{r_{j,\varepsilon}^{\frac{n}{2}} \mu_{k,\varepsilon}^{\frac{n}{2}-1}}{s_{j,k,\varepsilon}^{n-1}} \int_{B_{j,k,\varepsilon}} |w_{j,\varepsilon}| \, dv_{g_{j,\varepsilon}}\right) \\
&= O\left(\sum_{k\in\mathcal{D}_j} \frac{r_{j,\varepsilon}^{\frac{n}{2}+1} \mu_{k,\varepsilon}^{\frac{n}{2}-1}}{s_{j,k,\varepsilon}^{n}} \|w_{j,\varepsilon}\|_{2^\star} \mathrm{Vol}_{g_{j,\varepsilon}}\left(B_{j,k,\varepsilon}\right)^{\frac{2^\star-1}{2^\star}}\right) \\
&= O\left(\left(\sum_{k\in\mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{\frac{n}{2}-1}\right) \|\nabla w_{j,\varepsilon}\|_2\right).
\end{aligned}$$

In the same way, one also gets that

$$\begin{aligned}
&\int_{B_0(\delta_j)} |w_{j,\varepsilon}| \left|\Delta_{g_{j,\varepsilon}} \widetilde{\sigma}_{j,\varepsilon}\right| u_{j,\varepsilon} \, dv_{g_{j,\varepsilon}} \\
&= O\left(\left(\sum_{k\in\mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{\frac{n}{2}-1}\right) \|\nabla w_{j,\varepsilon}\|_2\right)
\end{aligned}$$

and that

$$\begin{aligned}
&\int_{B_0(\delta_j)} \left|\widetilde{\sigma}_{j,\varepsilon} - \widetilde{\sigma}_{j,\varepsilon}^{2^\star-1}\right| u_{j,\varepsilon}^{2^\star-1} |w_{j,\varepsilon}| \, dv_{g_{j,\varepsilon}} \\
&= O\left(\left(\sum_{k\in\mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{\frac{n}{2}+1}\right) \|\nabla w_{j,\varepsilon}\|_2\right) \\
&= o\left(\left(\sum_{k\in\mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{\frac{n}{2}-1}\right) \|\nabla w_{j,\varepsilon}\|_2\right)
\end{aligned}$$

since $\mu_{k,\varepsilon} = o\left(s_{j,k,\varepsilon}\right)$ by (2.9) and (2.10). At last, we write thanks to

(0.1), (2.33), (2.37), Hölder's and Sobolev's inequalities that

$$r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} h_{j,\varepsilon}\left(\eta_j \widetilde{v}_{j,\varepsilon}\right) w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} = O\left(r_{j,\varepsilon}^2 \|\nabla w_{j,\varepsilon}\|_2 \left(\int_{B_0(\delta_j)} \widetilde{\psi}_{j,\varepsilon}^{\frac{2^\star}{2^\star-1}}\, dv_{g_{j,\varepsilon}}\right)^{\frac{2^\star-1}{2^\star}}\right)$$

which leads with direct computations to

$$r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} h_{j,\varepsilon}\left(\eta_j \widetilde{v}_{j,\varepsilon}\right) w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} = o\big(\|\nabla w_{j,\varepsilon}\|_2^2\big) + o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + o\left(\nu_{j,\varepsilon}^3\right).$$

Plugging these estimates into (2.42), we obtain that

$$\begin{aligned}(2.43)\quad \big(1+o(1)\big) \int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|_{g_{j,\varepsilon}}^2\, dv_{g_{j,\varepsilon}} &= \int_{B_0(\delta_j)} \left(\eta_j \widetilde{v}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} + o\left(\nu_{j,\varepsilon}^3\right) \\ &\quad + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + O\left(\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{\frac{n}{2}-1}\right)\|\nabla w_{j,\varepsilon}\|_2\right).\end{aligned}$$

Relations (2.24) and (2.38) give that

$$J_{j,\varepsilon}\left(y_{j,\varepsilon}, \nu_{j,\varepsilon}, \theta_{j,\varepsilon}\right) = \int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|_{g_{j,\varepsilon}}^2\, dv_{g_{j,\varepsilon}} \to 0$$

as $\varepsilon \to 0$ so that Hölder's and Sobolev's inequalities permit us to write with (2.38) that

$$\begin{aligned}(2.44)\quad \int_{B_0(\delta_j)} \left(\eta_j \widetilde{v}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} &= \left(1+\theta_{j,\varepsilon}\right)^{2^\star-1} \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} \\ &\quad + \left(2^\star-1\right)\left(1+\theta_{j,\varepsilon}\right)^{2^\star-2} \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2} w_{j,\varepsilon}^2\, dv_{g_{j,\varepsilon}} \\ &\quad + o\left(\int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|_{g_{j,\varepsilon}}^2\, dv_{g_{j,\varepsilon}}\right).\end{aligned}$$

Direct computations give now thanks to (2.36) and to the Cartan expansion of the metric $g_{j,\varepsilon}$ around 0 that

$$\begin{aligned}
&\int_{B_0(\delta_j)} \left(\nabla w_{j,\varepsilon}, \nabla\left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} \\
&= \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} \\
&\quad + O\left(r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} |x|^2 \widetilde{\psi}_{j,\varepsilon}^{2^\star-1} |w_{j,\varepsilon}|\; dv_{g_{j,\varepsilon}}\right) \\
&\quad + O\left(r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} |x|^2 \left|\nabla \widetilde{\psi}_{j,\varepsilon}\right| |\nabla w_{j,\varepsilon}|\; dv_{g_{j,\varepsilon}}\right) + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right)
\end{aligned}$$

so that, using (2.39), Hölder's and Sobolev's inequalities, we get that

$$\begin{aligned}
&\int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} \\
&= O\left(r_{j,\varepsilon}^2 \|\nabla w_{j,\varepsilon}\|_2 \left(\int_{B_0(\delta_j)} \left(|x|^2 \widetilde{\psi}_{j,\varepsilon}^{2^\star-1}\right)^{\frac{2^\star}{2^\star-1}} dv_{g_{j,\varepsilon}}\right)^{\frac{2^\star-1}{2^\star}}\right) \\
&\quad + O\left(r_{j,\varepsilon}^2 \|\nabla w_{j,\varepsilon}\|_2 \left(\int_{B_0(\delta_j)} |x|^4 \left|\nabla \widetilde{\psi}_{j,\varepsilon}\right|_{g_{j,\varepsilon}}^2 dv_{g_{j,\varepsilon}}\right)^{\frac{1}{2}}\right) \\
&\quad + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) .
\end{aligned}$$

After simple computations, we finally obtain that

$$\begin{aligned}
&\int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} \\
&= O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + o\left(\nu_{j,\varepsilon}^3\right) + o\big(\|\nabla w_{j,\varepsilon}\|_2^2\big).
\end{aligned} \tag{2.45}$$

Coming back to (2.43) with (2.44) and (2.45), we arrive thanks to (2.31) to

$$(2.46)\quad \left(1+o\left(1\right)\right)\int_{B_0(\delta_j)}\left|\nabla w_{j,\varepsilon}\right|^2_{g_{j,\varepsilon}}\,dv_{g_{j,\varepsilon}} = \left(2^\star-1\right)\int_{B_0(\delta_j)}\left(\eta_j\widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2} w_{j,\varepsilon}^2\,dv_{g_{j,\varepsilon}} + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + O\left(\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{\frac{n}{2}-1}\right)\left\|\nabla w_{j,\varepsilon}\right\|_2\right) + o\left(\nu_{j,\varepsilon}^3\right).$$

Let us now consider the following eigenvalue problem:

$$(2.47)\quad \begin{cases} \Delta_{g_{j,\varepsilon}}\zeta_{i,\varepsilon} = \tau_{i,\varepsilon}\left(\eta_j\widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2}\zeta_{i,\varepsilon} \text{ in } B_0\left(\delta_j\right) \\ \zeta_{i,\varepsilon}=0 \text{ on } \partial B_0\left(\delta_j\right) \\ \int_{B_0(\delta_j)}\left(\eta_j\widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2}\zeta_{i,\varepsilon}\zeta_{k,\varepsilon}\,dv_{g_{j,\varepsilon}} = K_n^{-\frac{n}{2}}\delta_{ik}\end{cases}$$

with $\tau_{1,\varepsilon}\le\dots\le\tau_{i,\varepsilon}\le\dots$. By the result of Appendix 1, we know that

$$(2.48)\quad \lim_{\varepsilon\to 0}\tau_{i,\varepsilon}=\tau_i \text{ for all } i\in\mathbb{N}^\star$$

and that

$$(2.49)\quad \lim_{\varepsilon\to 0}\int_{B_0(\delta_j)}\left|\nabla\left(\zeta_{i,\varepsilon}-\widetilde{\zeta}_{i,\varepsilon}\right)\right|^2_{g_{j,\varepsilon}}\,dv_{g_{j,\varepsilon}}=0 \text{ for all } i\in\mathbb{N}^\star$$

where

$$(2.50)\quad \widetilde{\zeta}_{i,\varepsilon}=\left(\frac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}\right)^{\frac{n}{2}-1}\zeta_i\left(\frac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}x\right)$$

with (ζ_i,τ_i) the solutions of the following eigenvalue problem:

$$\begin{cases}\Delta_\xi\zeta_i=\tau_i u^{2^\star-2}\zeta_i \text{ in } \mathbb{R}^n \\ \int_{\mathbb{R}^n}u^{2^\star-2}\zeta_i\zeta_k\,dv_\xi = K_n^{-\frac{n}{2}}\delta_{ik}.\end{cases}$$

Thanks to the work of Bianchi-Egnell [3], we know that

$$\begin{aligned} &\zeta_1 = u\,, \quad \tau_1 = 1, \\ &\zeta_i = \lambda_i \frac{\partial u}{\partial x_{i-1}}, \quad \tau_i = 2^\star - 1 \text{ for } i = 2, \dots, n+1, \\ &\zeta_{n+2} = \lambda_{n+2}\left(u - \frac{2}{n(n-2)}|x|^2 u^{\frac{n}{n-2}}\right), \tau_{n+2} = 2^\star - 1, \end{aligned} \tag{2.51}$$

where $\lambda_2, \dots, \lambda_{n+2}$ are some positive real numbers and that

$$\tau_{n+3} > 2^\star - 1. \tag{2.52}$$

Let us now write that

$$w_{j,\varepsilon} = \sum_{i=1}^{n+2} \alpha_{i,\varepsilon}\zeta_{i,\varepsilon} + R_{j,\varepsilon}$$

with

$$\alpha_{i,\varepsilon} = \frac{\int_{B_0(\delta_j)} \left(\nabla w_{j,\varepsilon}, \nabla \zeta_{i,\varepsilon}\right)_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}}}{\int_{B_0(\delta_j)} |\nabla \zeta_{i,\varepsilon}|^2_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}}} \tag{2.53}$$

so that

$$\int_{B_0(\delta_j)} \left(\nabla R_{j,\varepsilon}, \nabla \zeta_{i,\varepsilon}\right)_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}} = 0$$

for $i = 1, \dots, n+2$. In particular, we obtain thanks to (2.48) that

$$\begin{aligned} &\int_{B_0(\delta_j)} |\nabla R_{j,\varepsilon}|^2_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}} \\ &\geq \left(\tau_{n+3} + o(1)\right) \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star - 2} R_{j,\varepsilon}^2 \, dv_{g_{j,\varepsilon}}. \end{aligned} \tag{2.54}$$

We also have that

$$\begin{aligned} &\int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|^2_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}} \\ &= K_n^{-\frac{n}{2}} \sum_{i=1}^{n+2} \tau_{i,\varepsilon}\alpha_{i,\varepsilon}^2 + \int_{B_0(\delta_j)} |\nabla R_{j,\varepsilon}|^2_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}} \end{aligned} \tag{2.55}$$

thanks to (2.47). At last, we can write that

$$\int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2} w_{j,\varepsilon}^2 \, dv_{g_{j,\varepsilon}} = K_n^{-\frac{n}{2}} \sum_{i=1}^{n+2} \alpha_{i,\varepsilon}^2 + \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2} R_{j,\varepsilon}^2 \, dv_{g_{j,\varepsilon}}. \tag{2.56}$$

We now estimate the $\alpha_{i,\varepsilon}$'s. We write thanks to (2.47), (2.49) and (2.53) that

$$\begin{aligned} K_n^{-\frac{n}{2}} \tau_{i,\varepsilon} \alpha_{i,\varepsilon} &= \int_{B_0(\delta_j)} \left(\nabla w_{j,\varepsilon}, \nabla \left(\zeta_{i,\varepsilon} - \widetilde{\zeta}_{i,\varepsilon}\right)\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} \\ &\quad + \int_{B_0(\delta_j)} \left(\nabla w_{j,\varepsilon}, \nabla \left(\widetilde{\zeta}_{i,\varepsilon}\right)\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} \\ &= \int_{B_0(\delta_j)} \left(\nabla w_{j,\varepsilon}, \nabla \left(\widetilde{\zeta}_{i,\varepsilon}\right)\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} + o\big(\|\nabla w_{j,\varepsilon}\|_2\big). \end{aligned}$$

It is then easily checked that

$$\int_{B_0(\delta_j)} \left(\nabla w_{j,\varepsilon}, \nabla \left(\widetilde{\zeta}_{i,\varepsilon}\right)\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} = O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right)$$

for $i = 1, \dots, n+2$ thanks to (2.39), (2.40), (2.41), (2.50) and (2.51). Thus we obtain that

$$\alpha_{i,\varepsilon}^2 = o\big(\|\nabla w_{j,\varepsilon}\|_2^2\big) + o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right).$$

Then (2.55) becomes

$$\|\nabla w_{j,\varepsilon}\|_2^2 \big(1 + o(1)\big) = o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + \|\nabla R_{j,\varepsilon}\|_2^2$$

and (2.56) becomes

$$\begin{aligned} \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2} w_{j,\varepsilon}^2 \, dv_{g_{j,\varepsilon}} &= \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2} R_{j,\varepsilon}^2 \, dv_{g_{j,\varepsilon}} \\ &\quad + o\big(\|\nabla w_{j,\varepsilon}\|_2^2\big) + o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right). \end{aligned}$$

Using (2.46), (2.52) and (2.54), we thus obtain that
(2.57)

$$\|\nabla w_{j,\varepsilon}\|_2^2 = O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + o\left(\nu_{j,\varepsilon}^3\right) + O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right).$$

We claim now that

$$\theta_{j,\varepsilon} = O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + o\big(\nu_{j,\varepsilon}\big) + O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right). \tag{2.58}$$

In order to prove this claim, we first note that

$$\int_{B_0(\delta_j)} \left|\nabla\left(\eta_j \widetilde{v}_{j,\varepsilon}\right)\right|^2_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} = \int_{B_0(\delta_j)} \left|\nabla w_{j,\varepsilon}\right|^2_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} + (1+\theta_{j,\varepsilon})^2 \int_{B_0(\delta_j)} \left|\nabla\left(\eta_j\widetilde{\psi}_{j,\varepsilon}\right)\right|^2_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}}$$

thanks to (2.38) and (2.39). Direct computations lead then with the Cartan expansion of the metric $g_{j,\varepsilon}$ around 0 to

$$\begin{aligned}
&\int_{B_0(\delta_j)} \left|\nabla\left(\eta_j\widetilde{\psi}_{j,\varepsilon}\right)\right|^2_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} \\
&= \int_{B_0(\delta_j)} \left|\nabla\widetilde{\psi}_{j,\varepsilon}\right|^2 dx + O\left(r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} |x|^2 \left|\nabla\widetilde{\psi}_{j,\varepsilon}\right|^2 dx\right) \\
&\quad + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) \\
&= K_n^{-\frac{n}{2}} + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + o\big(\nu_{j,\varepsilon}\big).
\end{aligned}$$

We thus get thanks to (2.31) and (2.57) that
(2.59)

$$\int_{B_0(\delta_j)} \left|\nabla\left(\eta_j \widetilde{v}_{j,\varepsilon}\right)\right|^2_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} = K_n^{-\frac{n}{2}}\left(1+\theta_{j,\varepsilon}\right)^2 + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right) + o\big(\nu_{j,\varepsilon}\big).$$

Independently, using (2.33), (2.35) and (2.36), we have that

$$\begin{aligned}
&\int_{B_0(\delta_j)} \left|\nabla\left(\eta_j \widetilde{v}_{j,\varepsilon}\right)\right|^2_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}} \\
&= \int_{B_0(\delta_j)} \left(\eta_j \widetilde{v}_{j,\varepsilon}\right)^{2^\star} \, dv_{g_{j,\varepsilon}} - r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} h_{j,\varepsilon}\left(\eta_j \widetilde{v}_{j,\varepsilon}\right)^2 \, dv_{g_{j,\varepsilon}} \\
&\quad + O\left(\int_{B_0(\delta_j)} \left|\nabla u_{j,\varepsilon}\right|_{g_{j,\varepsilon}} \left|\nabla \widetilde{\sigma}_{j,\varepsilon}\right|_{g_{j,\varepsilon}} u_{j,\varepsilon} \, dv_{g_{j,\varepsilon}}\right) + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) \\
&\quad + O\left(\int_{B_0(\delta_j)} \left|\Delta_{g_{j,\varepsilon}} \widetilde{\sigma}_{j,\varepsilon}\right| u_{j,\varepsilon}^2 \, dv_{g_{j,\varepsilon}}\right) \\
&\quad + O\left(\int_{B_0(\delta_j)} \left|\widetilde{\sigma}_{j,\varepsilon}^2 - \widetilde{\sigma}_{j,\varepsilon}^{2^\star}\right| u_{j,\varepsilon}^{2^\star} \, dv_{g_{j,\varepsilon}}\right).
\end{aligned}$$

Following what we did to estimate the different terms of (2.42), we deduce from this equation the following:

$$\begin{aligned}
&\int_{B_0(\delta_j)} \left|\nabla\left(\eta_j \widetilde{v}_{j,\varepsilon}\right)\right|^2_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}} \\
&= \int_{B_0(\delta_j)} \left(\eta_j \widetilde{v}_{j,\varepsilon}\right)^{2^\star} \, dv_{g_{j,\varepsilon}} - r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} h_{j,\varepsilon}\left(\eta_j \widetilde{v}_{j,\varepsilon}\right)^2 \, dv_{g_{j,\varepsilon}} \\
&\quad + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + O\left(\sum_{k\in\mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right).
\end{aligned}$$

Writing thanks to (2.38), Hölder's and Sobolev's inequalities that

$$\begin{aligned}
&\int_{B_0(\delta_j)} \left(\eta_j \widetilde{v}_{j,\varepsilon}\right)^{2^\star} \, dv_{g_{j,\varepsilon}} \\
&= \left(1+\theta_{j,\varepsilon}\right)^{2^\star} \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star} \, dv_{g_{j,\varepsilon}} \\
&\quad + 2^\star\left(1+\theta_{j,\varepsilon}\right)^{2^\star-1} \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon} \, dv_{g_{j,\varepsilon}} \\
&\quad + O\left(\int_{B_0(\delta_j)} \left|\nabla w_{j,\varepsilon}\right|^2_{g_{j,\varepsilon}} \, dv_{g_{j,\varepsilon}}\right)
\end{aligned}$$

and thanks to the Cartan expansion of the metric $g_{j,\varepsilon}$ around 0 that

$$\int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star} \, dv_{g_{j,\varepsilon}} = K_n^{-\frac{n}{2}} + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + o\left(\nu_{j,\varepsilon}\right),$$

we get thanks to (2.31), (2.45) and (2.57) that

$$\int_{B_0(\delta_j)} (\eta\widetilde{v}_{j,\varepsilon})^{2^\star}\, dv_{g_{j,\varepsilon}} = (1+\theta_{j,\varepsilon})^{2^\star} K_n^{-\frac{n}{2}} + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + o\left(\nu_{j,\varepsilon}\right) + O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right).$$

Using (2.37), it is easily checked that

$$r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} h_{j,\varepsilon}\left(\eta_j\widetilde{v}_{j,\varepsilon}\right)^2\, dv_{g_{j,\varepsilon}} = o\big(\nu_{j,\varepsilon}\big).$$

Thus we arrive to

$$\begin{aligned}&\int_{B_0(\delta_j)} \left|\nabla\left(\eta_j\widetilde{v}_{j,\varepsilon}\right)\right|^2_{g_{j,\varepsilon}}\, dv_{g_{j,\varepsilon}}\\ &= (1+\theta_{j,\varepsilon})^{2^\star} K_n^{-\frac{n}{2}} + O\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + o\big(\nu_{j,\varepsilon}\big)\\ &\quad + O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right).\end{aligned}$$

Combining (2.59) with this last relation, we get (2.58).

We now apply the Pohozaev identity with test function $f = \frac{1}{2}|x|^2$, to $u_{j,\varepsilon}$ in $\Omega_{j,\varepsilon}$ (see Appendix 2) where

$$\Omega_{j,\varepsilon} = B_0\left(\delta_j\right) \setminus \bigcup_{k\in\mathcal{D}_j} \Omega_{k,j,\varepsilon}$$

with

$$\Omega_{j,k,\varepsilon} = r_{j,\varepsilon}^{-1}\exp_{y_{j,\varepsilon}}^{-1}\left(B_{x_{k,\varepsilon}}\left(\frac{R_k}{2}s_{j,k,\varepsilon}\right)\right).$$

We thus have that

$$\begin{aligned}&\int_{\Omega_{j,\varepsilon}}\left(r_{j,\varepsilon}^2 h_{j,\varepsilon} + \frac{r_{j,\varepsilon}^2}{2}\left(\nabla h_{j,\varepsilon},\nabla f\right)_{g_{j,\varepsilon}} + \frac{1}{4}\left(\Delta_{g_{j,\varepsilon}}\right)^2 f\right) u_{j,\varepsilon}^2\, dv_{g_{j,\varepsilon}}\\ &= \frac{1}{n}\int_{\Omega_{j,\varepsilon}}\big(\Delta_{g_{j,\varepsilon}} f + n\big) u_{j,\varepsilon}^{2^\star}\, dv_{g_{j,\varepsilon}}\\ &\quad + \int_{\Omega_{j,\varepsilon}}\big(\nabla^2 f - g_{j,\varepsilon}\big)\big(\nabla u_{j,\varepsilon},\nabla u_{j,\varepsilon}\big)\, dv_{g_{j,\varepsilon}} + A_\varepsilon\end{aligned} \tag{2.60}$$

where A_ε is the boundary term

$$A_\varepsilon = \int_{\partial\Omega_{j,\varepsilon}} \left(\frac{1}{2} \left| \nabla u_{j,\varepsilon} \right|^2_{g_{j,\varepsilon}} \left(\nabla f, \nu \right)_{g_{j,\varepsilon}} - \left(\nabla u_{j,\varepsilon}, \nabla f \right)_{g_{j,\varepsilon}} \left(\nabla u_{j,\varepsilon}, \nu \right)_{g_{j,\varepsilon}} \right) d\sigma_{g_{j,\varepsilon}}$$
$$- \frac{n-2}{2} \int_{\partial\Omega_{j,\varepsilon}} u_{j,\varepsilon} \left(\nabla u_{j,\varepsilon}, \nu \right)_{g_{j,\varepsilon}} d\sigma_{g_{j,\varepsilon}}$$
$$- \int_{\partial\Omega_{j,\varepsilon}} \left(\nabla f, \nu \right)_{g_{j,\varepsilon}} \left(\frac{1}{2^\star} u_{j,\varepsilon}^{2^\star} - \frac{r_{j,\varepsilon}^2}{2} h_{j,\varepsilon} u_{j,\varepsilon}^2 \right) d\sigma_{g_{j,\varepsilon}}$$
$$+ \frac{1}{2} \int_{\partial\Omega_{j,\varepsilon}} \left(\Delta_{g_{j,\varepsilon}} f + n \right) u_{j,\varepsilon} \left(\nabla u_{j,\varepsilon}, \nu \right)_{g_{j,\varepsilon}} d\sigma_{g_{j,\varepsilon}}$$
$$- \frac{1}{4} \int_{\partial\Omega_{j,\varepsilon}} \left(\nabla \left(\Delta_{g_{j,\varepsilon}} f \right), \nu \right)_{g_{j,\varepsilon}} u_{j,\varepsilon}^2 \, d\sigma_{g_{j,\varepsilon}}$$

where ν denotes the unit outer normal of $\partial\Omega_{j,\varepsilon}$. Note that, by (2.11) and (2.12),

$$\partial\Omega_{j,\varepsilon} = \cup_{k \in \mathcal{D}_j} \partial\Omega_{j,k,\varepsilon} \cup \partial B_0 \left(\delta_j \right)$$

and the union is a disjoint one. We denote by $A_{k,\varepsilon}$ the part of A_ε corresponding to $\partial\Omega_{j,k,\varepsilon}$. Noting that, on $\partial\Omega_{j,k,\varepsilon}$,

$$r_{j,\varepsilon} |\nabla f| \le d_g \left(y_{j,\varepsilon}, x_{k,\varepsilon} \right) + \frac{R_k}{2} s_{j,k,\varepsilon},$$

we can estimate $A_{k,\varepsilon}$ thanks to (0.1), (2.4), (2.10), (2.11), (2.14) and (2.34). This leads to

$$A_{k,\varepsilon} = O \left(\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}} \right)^{n-2} \left(1 + \frac{d_g \left(y_{j,\varepsilon}, x_{k,\varepsilon} \right)}{s_{j,k,\varepsilon}} \right) \right)$$

for all $k \in \mathcal{D}_j$. In order to estimate the part of A_ε corresponding to $\partial B_0 \left(\delta_j \right)$, we use (0.1), (2.3), (2.4), (2.31), (2.34), (2.36) and the explicit form of h_j given in Claim 2, point (i). We finally obtain that

$$A_\varepsilon = \frac{(n-2)^2}{2} a_n^{2-n} \omega_{n-1} h_j(0) \left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}} \right)^{n-2} + o \left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}} \right)^{n-2} \right)$$
$$+ O \left(\sum_{k \in \mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}} \right)^{n-2} \left(1 + \frac{d_g \left(y_{j,\varepsilon}, x_{k,\varepsilon} \right)}{s_{j,k,\varepsilon}} \right) \right). \tag{2.61}$$

Since $(\eta_j \widetilde{\psi}_{j,\varepsilon})$ is radially symmetrical, noting that $\widetilde{\sigma}_{j,\varepsilon} = 1$ in $\Omega_{j,\varepsilon}$, we get with (2.31), (2.34), (2.36), (2.38) and (2.57) that

$$\begin{aligned} &\int_{\Omega_{j,\varepsilon}} \left(\nabla^2 f - g_{j,\varepsilon}\right)\left(\nabla u_{j,\varepsilon}, \nabla u_{j,\varepsilon}\right) dv_{g_{j,\varepsilon}} \\ &\quad = o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + o\left(\nu_{j,\varepsilon}^3\right) + o\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right). \end{aligned} \tag{2.62}$$

The Cartan expansion of the metric $g_{j,\varepsilon}$ around 0 gives that

$$\begin{aligned} \Delta_{g_{j,\varepsilon}} f + n = \frac{1}{3}\mathrm{Ric}_{g_{j,\varepsilon}}(0)_{kl}\, x^k x^l & \\ + \frac{1}{2}\partial_{klm}\left(\ln\sqrt{|g_{j,\varepsilon}|}\right)(0) x^k x^l x^m + O\left(r_{j,\varepsilon}^4 |x|^4\right) & \end{aligned}$$

in $B_0(\delta_j)$ where $\mathrm{Ric}_{g_{j,\varepsilon}}$ denotes the Ricci curvature of $g_{j,\varepsilon}$ and where $|g_{j,\varepsilon}|$ is the determinant of the matrix $\left((g_{j,\varepsilon})_{ik}\right)$. Since $\widetilde{\sigma}_{j,\varepsilon} = 1$ in $\Omega_{j,\varepsilon}$, we can write thanks to (2.3), (2.31), (2.36), (2.38) and to the fact that $\left(\widetilde{\psi}_{j,\varepsilon}\right)$ is radially symmetrical that

$$\begin{aligned} &\int_{\Omega_{j,\varepsilon}} (\Delta_{g_{j,\varepsilon}} f + n) u_{j,\varepsilon}^{2^\star}\, dv_{g_{j,\varepsilon}} \\ &= \frac{(1+\theta_{j,\varepsilon})^{2^\star}}{3}\mathrm{Ric}_{g_{j,\varepsilon}}(0)_{kl}\int_{B_0(\delta_j)} x^k x^l \widetilde{\psi}_{j,\varepsilon}^{2^\star}\, dx + o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) \\ &\quad + O\left(r_{j,\varepsilon}^2 \int_{B_0(\delta_j)} |x|^2 \widetilde{\psi}_{j,\varepsilon}^{2^\star-1} |w_{j,\varepsilon}|\, dx\right) + O\left(r_{j,\varepsilon}^4 \int_{B_0(\delta_j)} |x|^4 \widetilde{\psi}_{j,\varepsilon}^{2^\star}\, dx\right) \\ &\quad + O\left(r_{j,\varepsilon}^2 \sum_{k\in\mathcal{D}_j}\int_{\Omega_{j,k,\varepsilon}} |x|^2 \widetilde{\psi}_{j,\varepsilon}^{2^\star}\, dv_{g_{j,\varepsilon}}\right) + O\left(r_{j,\varepsilon}^2 \|w_{j,\varepsilon}\|_{2^\star}^{2^\star}\right). \end{aligned}$$

Using (2.57) and (2.58), this leads thanks to Hölder's and Sobolev's inequalities and to some explicit computations to the following:

(2.63)

$$\begin{aligned}&\int_{\Omega_{j,\varepsilon}}\left(\Delta_{g_{j,\varepsilon}} f+n\right)u_{j,\varepsilon}^{2^\star}\,dv_{g_{j,\varepsilon}}\\&=\frac{n}{3}K_n^{-\frac{n}{2}}S_g\left(y_{j,\varepsilon}\right)\nu_{j,\varepsilon}^2+o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right)+o\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right)\\&\quad+O\left(r_{j,\varepsilon}^2\sum_{k\in\mathcal{D}_j}\int_{\Omega_{j,k,\varepsilon}}|x|^2\widetilde{\psi}_{j,\varepsilon}^{2^\star}\,dv_{g_{j,\varepsilon}}\right)+o\left(\nu_{j,\varepsilon}^3\right).\end{aligned}$$

Let $k\in\mathcal{D}_j$. We write that

$$\begin{aligned}&r_{j,\varepsilon}^2\int_{\Omega_{j,k,\varepsilon}}|x|^2\widetilde{\psi}_{j,\varepsilon}^{2^\star}\,dv_{g_{j,\varepsilon}}\\&=\int_{B_{x_{k,\varepsilon}}\left(\frac{R_k}{2}s_{j,k,\varepsilon}\right)}d_g\left(y_{j,\varepsilon},x\right)^2\left(\frac{\nu_{j,\varepsilon}}{\nu_{j,\varepsilon}^2+a_nd_g\left(y_{j,\varepsilon},x\right)^2}\right)^n\,dv_g.\end{aligned}$$

Let us assume first that $d_g\left(y_{j,\varepsilon},x_{k,\varepsilon}\right)=O\left(s_{j,k,\varepsilon}\right)$. Then we write that for some $R>0$ large and for ε small,

$$\begin{aligned}&r_{j,\varepsilon}^2\int_{\Omega_{j,k,\varepsilon}}|x|^2\widetilde{\psi}_{j,\varepsilon}^{2^\star}\,dv_{g_{j,\varepsilon}}\\&\le\int_{B_{y_{j,\varepsilon}}\left(Rs_{j,k,\varepsilon}\right)}d_g\left(y_{j,\varepsilon},x\right)^2\left(\frac{\nu_{j,\varepsilon}}{\nu_{j,\varepsilon}^2+a_nd_g\left(y_{j,\varepsilon},x\right)^2}\right)^n\,dv_g\,.\end{aligned}$$

Independently, by (1.1), (2.9) and (2.10), we have that $s_{j,k,\varepsilon}=o\big(d_g(x_{j,\varepsilon},x_{k,\varepsilon})\big)$. Thanks to (2.31), we then obtain that $s_{j,k,\varepsilon}=o\left(\nu_{j,\varepsilon}\right)$. Thus we get that

$$\begin{aligned}r_{j,\varepsilon}^2\int_{\Omega_{j,k,\varepsilon}}|x|^2\widetilde{\psi}_{j,\varepsilon}^{2^\star}\,dv_{g_{j,\varepsilon}}&=O\left(\frac{s_{j,k,\varepsilon}^{n+2}}{\nu_{j,\varepsilon}^n}\right)\\&=O\left(\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\frac{s_{j,k,\varepsilon}^{2n}}{\nu_{j,\varepsilon}^n\mu_{k,\varepsilon}^{n-2}}\right).\end{aligned}$$

Since we assumed that $d_g\left(y_{j,\varepsilon},x_{k,\varepsilon}\right)=O\left(s_{j,k,\varepsilon}\right)$, we also get thanks to (2.9), (2.10) and (2.31) that $s_{j,k,\varepsilon}^2=O\left(\nu_{j,\varepsilon}\mu_{k,\varepsilon}\right)$. Thus, since $\mu_{k,\varepsilon}\to 0$

as $\varepsilon \to 0$, we have obtained in this case that

$$r_{j,\varepsilon}^2 \int_{\Omega_{j,k,\varepsilon}} |x|^2 \widetilde{\psi}_{j,\varepsilon}^{2^\star} \, dv_{g_{j,\varepsilon}} = o\left(\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right).$$

Let us assume now that $\frac{d_g(y_{j,\varepsilon}, x_{k,\varepsilon})}{s_{j,k,\varepsilon}} \to +\infty$ as $\varepsilon \to 0$. In this case, we can write that

$$\begin{aligned} & r_{j,\varepsilon}^2 \int_{\Omega_{j,k,\varepsilon}} |x|^2 \widetilde{\psi}_{j,\varepsilon}^{2^\star} \, dv_{g_{j,\varepsilon}} \\ & = O\left(d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)^2 \left(\frac{s_{j,k,\varepsilon}}{\nu_{j,\varepsilon}}\right)^n \left(1 + a_n \frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)^2}{\nu_{j,\varepsilon}^2}\right)^{-n}\right). \end{aligned}$$

We write then thanks to (1.1), (2.10) and (2.31) that

$$\left(\frac{s_{j,k,\varepsilon}}{\nu_{j,\varepsilon}}\right)^n = O\left(\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2} \frac{\mu_{k,\varepsilon}}{\nu_{j,\varepsilon}} \left(1 + a_n \frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)^2}{\nu_{j,\varepsilon}^2}\right)^{n-1}\right)$$

so that we get that

$$\begin{aligned} & r_{j,\varepsilon}^2 \int_{\Omega_{j,k,\varepsilon}} |x|^2 \widetilde{\psi}_{j,\varepsilon}^{2^\star} \, dv_{g_{j,\varepsilon}} \\ & = O\left(\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2} \mu_{k,\varepsilon}\nu_{j,\varepsilon} \frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)^2}{\nu_{j,\varepsilon}^2 + a_n d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)^2}\right). \end{aligned}$$

This gives that the estimate of the first case also holds in this second case. We have thus obtained that

$$r_{j,\varepsilon}^2 \int_{\Omega_{j,k,\varepsilon}} |x|^2 \widetilde{\psi}_{j,\varepsilon}^{2^\star} \, dv_{g_{j,\varepsilon}} = o\left(\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right) \tag{2.64}$$

for all $k \in \mathcal{D}_j$. Let us estimate the left-hand side term of (2.60). Using (0.1) and the Cartan expansion of the metric $g_{j,\varepsilon}$ around 0, one gets that

$$\begin{aligned} & r_{j,\varepsilon}^2 h_{j,\varepsilon} + \frac{r_{j,\varepsilon}^2}{2} \left(\nabla h_{j,\varepsilon}, \nabla f\right)_{g_{j,\varepsilon}} + \frac{1}{4}\left(\Delta_{g_{j,\varepsilon}}\right)^2 f \\ & = r_{j,\varepsilon}^2 \left(h_\varepsilon\left(y_{j,\varepsilon}\right) - \frac{1}{6} S_g\left(y_{j,\varepsilon}\right)\right) \\ & \quad + \left(\frac{3}{2} r_{j,\varepsilon}^2 \partial_k h_{j,\varepsilon}(0) + \frac{1}{4}\partial_k \left(\Delta_{g_{j,\varepsilon}}^2 f\right)(0)\right) x^k + O\left(r_{j,\varepsilon}^4 |x|^2\right) \end{aligned}$$

in $B_0(\delta_j)$ so that we obtain with (2.4), (2.31), (2.36), (2.38) and the fact that $\left(\widetilde{\psi}_{j,\varepsilon}\right)$ is radially symmetrical that

$$\begin{aligned}
&\int_{\Omega_{j,\varepsilon}}\left(r_{j,\varepsilon}^2 h_{j,\varepsilon}+\frac{r_{j,\varepsilon}^2}{2}\left(\nabla h_{j,\varepsilon},\nabla f\right)_{g_{j,\varepsilon}}+\frac{1}{4}\left(\Delta_{g_{j,\varepsilon}}\right)^2 f\right)u_{j,\varepsilon}^2\,dv_{g_{j,\varepsilon}}\\
&=r_{j,\varepsilon}^2\left(h_\varepsilon\left(y_{j,\varepsilon}\right)-\frac{1}{6}S_g\left(y_{j,\varepsilon}\right)\right)\left(1+\theta_{j,\varepsilon}\right)^2\int_{B_0(\delta_j)}\widetilde{\psi}_{j,\varepsilon}^2\,dx\\
&\quad+o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right)\\
&\quad+O\left(r_{j,\varepsilon}^2\sum_{k\in\mathcal{D}_j}\int_{\Omega_{j,k,\varepsilon}}\widetilde{\psi}_{j,\varepsilon}^2\,dv_{g_{j,\varepsilon}}\right)+O\left(r_{j,\varepsilon}^4\int_{B_0(\delta_j)}|x|^2\widetilde{\psi}_{j,\varepsilon}^2\,dx\right)\\
&\quad+O\left(r_{j,\varepsilon}^2\int_{B_0(\delta_j)}\widetilde{\psi}_{j,\varepsilon}|w_{j,\varepsilon}|\,dx\right)+O\left(r_{j,\varepsilon}^2\int_{B_0(\delta_j)}|w_{j,\varepsilon}|^2\,dx\right).
\end{aligned}$$

This leads thanks to (2.4), (2.57) and (2.58) and after computations similar to those developed above to prove (2.63) and (2.64) to

$$\begin{aligned}
&\int_{\Omega_{j,\varepsilon}}\left(r_{j,\varepsilon}^2 h_{j,\varepsilon}+\frac{r_{j,\varepsilon}^2}{2}\left(\nabla h_{j,\varepsilon},\nabla f\right)_{g_{j,\varepsilon}}+\frac{1}{4}\left(\Delta_{g_{j,\varepsilon}}\right)^2 f\right)u_{j,\varepsilon}^2\,dv_{g_{j,\varepsilon}}\\
&=\left(h_\varepsilon\left(y_{j,\varepsilon}\right)-\frac{1}{6}S_g\left(y_{j,\varepsilon}\right)\right)\times\begin{cases}64\omega_3\nu_{j,\varepsilon}^2\ln\left(\dfrac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}\right) & \text{if } n=4\\[2ex] \dfrac{4(n-1)}{n-4}K_n^{-\frac{n}{2}}\nu_{j,\varepsilon}^2 & \text{if } n\geq 5\end{cases}\\
&\quad+o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right)+o\left(\nu_{j,\varepsilon}^3\right)+O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right).
\end{aligned}$$

Coming back to (2.60) with (2.61) – (2.64) and this last estimate, we finally arrive to

(2.65)

$$\frac{3}{2}\omega_2 h_j(0)\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}=o\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)+O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)\left(1+\frac{d_g\left(y_{j,\varepsilon},x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)\right)$$

when $n = 3$, to

(2.66)

$$\left(h_\varepsilon\left(y_{j,\varepsilon}\right) - C(4) S_g\left(y_{j,\varepsilon}\right)\right) r_{j,\varepsilon}^2 \ln\left(\frac{1}{\nu_{j,\varepsilon}}\right) = 2h_j(0) + o(1) + O\left(\frac{r_{j,\varepsilon}^2}{\nu_{j,\varepsilon}^2} \sum_{k \in \mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^2 \left(1 + \frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)\right)$$

when $n = 4$, to

$$\begin{aligned}\left(h_\varepsilon\left(y_{j,\varepsilon}\right) - C(5) S_g\left(y_{j,\varepsilon}\right)\right) r_{j,\varepsilon}^3 \nu_{j,\varepsilon}^{-1} &= 9\sqrt{15}\frac{\omega_4}{\omega_5} h_j(0) + o(1) \\ &\quad + O\left(\frac{r_{j,\varepsilon}^3}{\nu_{j,\varepsilon}^3} \sum_{k \in \mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^3 \left(1 + \frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)\right)\end{aligned} \quad (2.67)$$

when $n = 5$ and to

(2.68)

$$\begin{aligned}&\left(h_\varepsilon\left(y_{j,\varepsilon}\right) - C(n) S_g\left(y_{j,\varepsilon}\right)\right) r_{j,\varepsilon}^{n-2} \nu_{j,\varepsilon}^{4-n} \\ &= \frac{n-4}{4(n-1)} \frac{(n-2)^2}{2} a_n^{2-n} K_n^{\frac{n}{2}} \omega_{n-1} h_j(0) + o(1) + o\left(r_{j,\varepsilon}^{n-2} \nu_{j,\varepsilon}^{5-n}\right) \\ &\quad + O\left(\left(\frac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}\right)^{n-2} \sum_{k \in \mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2} \left(1 + \frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)\right)\end{aligned}$$

when $n \geq 6$. In these relations, $C(n) = \frac{n-2}{4(n-1)}$.

We prove now point (ii) of Claim 2 by induction on j. Let us start by $j = 1$. Assume that $r_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$. Note that, by (1.6), $\mathcal{A}_1 = \{2, \ldots, N\}$. We write with (2.2) that

$$r_{1,\varepsilon}^2 \leq \frac{\mu_{1,\varepsilon}}{\mu_{k,\varepsilon}} d_g\left(x_{1,\varepsilon}, x_{k,\varepsilon}\right)^2 + \mu_{1,\varepsilon}\mu_{k,\varepsilon}$$

for all $k \in \{2, \ldots, N\}$ which gives thanks to (1.1) and (1.6) that $d_g(x_{1,\varepsilon}, x_{k,\varepsilon}) \geq r_{1,\varepsilon}(1 + o(1))$ for all $k \in \{2, \ldots, N\}$. Then $\mathcal{C}_1 = \emptyset$ where $\mathcal{C}_1$ is as in (2.8) and thus $\mathcal{D}_1 = \emptyset$. Then (2.65) leads to a contradiction when

$n=3$ since $h_1(0)>0$ by (2.7) and $\frac{\nu_{1,\varepsilon}}{r_{1,\varepsilon}}\to 0$ as $\varepsilon\to 0$ thanks to (2.3) and (2.31). Thus, in dimension $n=3$, we have that $\liminf_{\varepsilon\to 0} r_{1,\varepsilon}>0$. This proves that the assertion of Claim 2 in the case $n=3$ holds for $j=1$. For $n\geq 4$, since $\mathcal{D}_1=\emptyset$, (2.66) – (2.68) clearly lead thanks to (0.1) and (2.31) to the assertion (ii) of Claim 2 for $j=1$.

Let now $j\in\left\{2,\dots,N\right\}$ and assume that the assertion of Claim 2 for $n=3$ holds for $i=1,\dots,j-1$ and that point (ii) of Claim 2 holds for $i=1,\dots,j-1$. Assume moreover that $r_{j,\varepsilon}\to 0$ as $\varepsilon\to 0$. Let $k\in\mathcal{D}_j$. Thanks to (1.6), (2.1) and (2.9), we have that $k<j$ and that $j\in\mathcal{A}_k$. Then, thanks to (2.2) and (2.10), we obtain that $s_{j,k,\varepsilon}\geq r_{k,\varepsilon}$. By (2.11), since we assumed that $r_{j,\varepsilon}\to 0$ as $\varepsilon\to 0$, we get that $r_{k,\varepsilon}\to 0$ as $\varepsilon\to 0$. If $n=3$, since $k<j$, the induction hypothesis gives a contradiction so that we get that $\mathcal{D}_j=\emptyset$. As above, we deduce then from (2.65) that the assertion of Claim 2 in the case $n=3$ holds for j. Assume now that $n\geq 4$. Let $k\in\mathcal{D}_j$. We write with (2.10) and (2.31) that

$$\frac{d_g\left(y_{j,\varepsilon},x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}=\frac{d_g\left(x_{j,\varepsilon},x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}+o\left(\frac{\mu_{j,\varepsilon}}{s_{j,k,\varepsilon}}\right)=O\left(\sqrt{\frac{\mu_{j,\varepsilon}}{\mu_{k,\varepsilon}}}\right).$$

Thus, since $s_{j,k,\varepsilon}\geq r_{k,\varepsilon}$ and thanks to (2.9),

$$\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\left(1+\frac{d_g\left(y_{j,\varepsilon},x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)=O\left(\sqrt{\frac{\mu_{j,\varepsilon}}{\mu_{k,\varepsilon}}}\left(\frac{\mu_{k,\varepsilon}}{r_{k,\varepsilon}}\right)^{n-2}\right).$$

Since $k<j$ and $r_{k,\varepsilon}\to 0$ as $\varepsilon\to 0$, we can apply point (ii) of Claim 2 to k to obtain that

$$\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\left(1+\frac{d_g\left(y_{j,\varepsilon},x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)=\begin{cases}O\left(\sqrt{\frac{\mu_{j,\varepsilon}}{\mu_{k,\varepsilon}}}\mu_{k,\varepsilon}^2\ln\left(\frac{1}{\mu_{k,\varepsilon}}\right)\right) & \text{if } n=4\\ O\left(\sqrt{\frac{\mu_{j,\varepsilon}}{\mu_{k,\varepsilon}}}\mu_{k,\varepsilon}^2\right) & \text{if } n\geq 5.\end{cases}$$

Thanks to (2.9), this leads to

$$\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\left(1+\frac{d_g\left(y_{j,\varepsilon},x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)=\begin{cases}o\left(\mu_{j,\varepsilon}^2\ln\left(\frac{1}{\mu_{j,\varepsilon}}\right)\right) & \text{if } n=4\\ o\left(\mu_{j,\varepsilon}^2\right) & \text{if } n\geq 5\end{cases}$$

for all $k\in\mathcal{D}_j$. Then (2.66) – (2.68) together with (0.1), (2.31) and this last estimate clearly lead to point (ii) of Claim 2 for j. This proves that the assertion of Claim 2 for $n = 3$ holds for all $j \in \{1, \dots, N\}$ and that point (ii) of Claim 2 holds for all $j \in \{1, \dots, N\}$ when $n \geq 4$.

It remains to prove point (iii) of Claim 2. We assume throughout the end of this section that $n \geq 4$. Let $j \in \{1, \dots, N\}$ and assume that $h_0(\overline{x}_j) > C(n) S_g(\overline{x}_j)$. Then, thanks to point (ii) of Claim 2, we have that

$$r_{j,\varepsilon} = O\left(\ln\left(\frac{1}{\mu_{j,\varepsilon}}\right)^{-\frac{1}{2}}\right) \text{ if } n = 4, \qquad r_{j,\varepsilon} = O\left(\mu_{j,\varepsilon}^{-\frac{n-4}{n-2}}\right) \text{ if } n \geq 5. \tag{2.69}$$

We let $0 < \delta < \delta_j$ and we set

$$\Omega_{j,\varepsilon}(\delta) = B_0(\delta) \setminus \cup_{k \in \mathcal{D}_j} \Omega_{j,k,\varepsilon}.$$

By integration by parts, we have that

$$\int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon} \Delta_\xi u_{j,\varepsilon}\, dx = \frac{1}{2} \int_{\partial\Omega_{j,\varepsilon}(\delta)} |\nabla u_{j,\varepsilon}|_\xi^2 \frac{x_\gamma}{|x|}\, d\sigma_\xi - \int_{\partial\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon} \frac{x^i \partial_i u_{j,\varepsilon}}{|x|}\, d\sigma_\xi \tag{2.70}$$

for all $\gamma = 1, \dots, n$. By (2.11) and (2.12), we have that

$$\partial\Omega_{j,\varepsilon}(\delta) = \cup_{k \in \mathcal{D}_j} \partial\Omega_{j,k,\varepsilon} \cup \partial B_0(\delta),$$

the union being disjoint. Using (2.36) and the explicit form of h_j, we get that

$$\lim_{\varepsilon \to 0} \left(\left(\frac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}\right)^{n-2} \int_{\partial B_0(\delta)} \left(\frac{1}{2} |\nabla u_{j,\varepsilon}|_\xi^2 \frac{x_\gamma}{|x|} - \partial_\gamma u_{j,\varepsilon} \frac{x^i \partial_i u_{j,\varepsilon}}{|x|} \right) d\sigma_\xi \right) = a_n^{2-n} (n-2)\, \omega_{n-1} \partial_\gamma h_j(0). \tag{2.71}$$

Let $k \in \mathcal{D}_j$. Thanks to (1.6), (2.1), (2.2), (2.9) and (2.10), we get successively that $k < j$, that $j \in \mathcal{A}_k$, that $r_{k,\varepsilon} \leq s_{j,k,\varepsilon}$ and that $r_{k,\varepsilon} \to 0$ as $\varepsilon \to 0$. Applying point (ii) of Claim 2 to k, we thus obtain that

$$\frac{1}{s_{j,k,\varepsilon}} = \begin{cases} O\left(\ln\left(\frac{1}{\mu_{k,\varepsilon}}\right)^{\frac{1}{2}}\right) & \text{if } n = 4 \\ O\left(\mu_{k,\varepsilon}^{-\frac{n-4}{n-2}}\right) & \text{if } n \geq 5 \end{cases} \tag{2.72}$$

for all $k \in \mathcal{D}_j$. Thanks to (2.14), we get that

$$\left(\frac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}\right)^{n-2} \int_{\partial \Omega_{j,k,\varepsilon}} |\nabla u_{j,\varepsilon}|_\xi^2 \, d\sigma_\xi = O\left(\frac{r_{j,\varepsilon}^{n-1}}{\nu_{j,\varepsilon}^{n-2}} \frac{\mu_{k,\varepsilon}^{n-2}}{s_{j,k,\varepsilon}^{n-1}}\right).$$

for all $k \in \mathcal{D}_j$ which gives thanks to (2.9), (2.31), (2.69) and (2.72) that

$$\left(\frac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}\right)^{n-2} \int_{\partial \Omega_{j,k,\varepsilon}} |\nabla u_{j,\varepsilon}|_\xi^2 \, d\sigma_\xi = o(1).$$

Coming back to (2.70) with (2.71) and this last estimate, we arrive to

$$\begin{aligned} &\lim_{\varepsilon \to 0} \left(\left(\frac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}\right)^{n-2} \int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon} \Delta_\xi u_{j,\varepsilon} \, dx\right) \\ &= a_n^{2-n} (n-2) \omega_{n-1} \partial_\gamma h_j(0). \end{aligned} \tag{2.73}$$

We write now that

$$\begin{aligned} \int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon} \Delta_\xi u_{j,\varepsilon} \, dx = &\int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon} \Delta_{g_{j,\varepsilon}} u_{j,\varepsilon} \, dx \\ &+ \int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon} \left(\Delta_\xi u_{j,\varepsilon} - \Delta_{g_{j,\varepsilon}} u_{j,\varepsilon}\right) dx. \end{aligned} \tag{2.74}$$

By Equation (2.35) and some integration by parts, we obtain that

$$\int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon} \Delta_{g_{j,\varepsilon}} u_{j,\varepsilon} \, dx = \frac{r_{j,\varepsilon}^2}{2} \int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma h_{j,\varepsilon} u_{j,\varepsilon}^2 \, dx + o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right)$$

using (0.1), (2.3), (2.4), (2.9), (2.14), (2.31), (2.36), (2.69) and (2.72) to estimate the different boundary terms. Noting that $\widetilde{\sigma}_{j,\varepsilon} = 1$ in $\Omega_{j,\varepsilon}(\delta)$, we get with (0.1), (2.33) and (2.37) that

$$\frac{r_{j,\varepsilon}^2}{2} \int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma h_{j,\varepsilon} u_{j,\varepsilon}^2 \, dx = O\left(r_{j,\varepsilon}^3 \int_{B_0(\delta)} \widetilde{\psi}_{j,\varepsilon}^2 \, dx\right) = o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right)$$

thanks to (2.31) and (2.69) so that we arrive to

$$\int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon} \Delta_{g_{j,\varepsilon}} u_{j,\varepsilon} \, dx = o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right). \tag{2.75}$$

We write that

$$\begin{aligned}(2.76)\qquad &\int_{\Omega_{j,\varepsilon}(\delta)}\partial_\gamma u_{j,\varepsilon}\left(\Delta_\xi u_{j,\varepsilon}-\Delta_{g_{j,\varepsilon}}u_{j,\varepsilon}\right)\,dx\\
&=\int_{\Omega_{j,\varepsilon}(\delta)}\partial_\gamma u_{j,\varepsilon}\left(g^{\alpha\beta}_{j,\varepsilon}-\delta^{\alpha\beta}\right)\partial_{\alpha\beta}u_{j,\varepsilon}\,dx\\
&\quad+\int_{\Omega_{j,\varepsilon}(\delta)}\frac{\partial_\alpha\left(\sqrt{|g_{j,\varepsilon}|}g^{\alpha\beta}_{j,\varepsilon}\right)}{\sqrt{|g_{j,\varepsilon}|}}\partial_\beta u_{j,\varepsilon}\partial_\gamma u_{j,\varepsilon}\,dx.\end{aligned}$$

By integration by parts, we get that

$$\begin{aligned}&\int_{\Omega_{j,\varepsilon}(\delta)}\partial_\gamma u_{j,\varepsilon}\left(g^{\alpha\beta}_{j,\varepsilon}-\delta^{\alpha\beta}\right)\partial_{\alpha\beta}u_{j,\varepsilon}\,dx\\
&=\frac{1}{2}\int_{\Omega_{j,\varepsilon}(\delta)}\partial_\gamma\left(g^{\alpha\beta}_{j,\varepsilon}\right)\partial_\alpha u_{j,\varepsilon}\partial_\beta u_{j,\varepsilon}\,dx\\
&\quad-\int_{\Omega_{j,\varepsilon}(\delta)}\partial_\alpha\left(g^{\alpha\beta}_{j,\varepsilon}\right)\partial_\gamma u_{j,\varepsilon}\partial_\beta u_{j,\varepsilon}\,dx+o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right)\end{aligned}$$

using (0.1), (2.3), (2.4), (2.9), (2.14), (2.31), (2.36), (2.69) and (2.72) to estimate the boundary terms. Thus we arrive with (2.76) to

$$\begin{aligned}&\int_{\Omega_{j,\varepsilon}(\delta)}\partial_\gamma u_{j,\varepsilon}\left(\Delta_\xi u_{j,\varepsilon}-\Delta_{g_{j,\varepsilon}}u_{j,\varepsilon}\right)\,dx\\
&=\frac{1}{2}\int_{\Omega_{j,\varepsilon}(\delta)}\partial_\gamma\left(g^{\alpha\beta}_{j,\varepsilon}\right)\partial_\alpha u_{j,\varepsilon}\partial_\beta u_{j,\varepsilon}\,dx+o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right)\\
&\quad+\int_{\Omega_{j,\varepsilon}(\delta)}g^{\alpha\beta}_{j,\varepsilon}\frac{\partial_\alpha\left(\sqrt{|g_{j,\varepsilon}|}\right)}{\sqrt{|g_{j,\varepsilon}|}}\partial_\beta u_{j,\varepsilon}\partial_\gamma u_{j,\varepsilon}\,dx.\end{aligned}$$

With an expansion of the different terms involving the metric $g_{j,\varepsilon}$ around 0, noting that

$$\int_{\partial B_0(r)}x^\alpha x^\beta x^\gamma d\sigma_\xi=0$$

for all α, β, γ and all $r>0$ and writing thanks to (2.38) that $u_{j,\varepsilon}=(1+\theta_{j,\varepsilon})\,\widetilde{\psi}_{j,\varepsilon}+w_{j,\varepsilon}$ in $\Omega_{j,\varepsilon}(\delta)$ for $\delta>0$ small enough since $\widetilde{\sigma}_{j,\varepsilon}=1$ in

$\Omega_{j,\varepsilon}(\delta)$, we obtain that

$$\begin{aligned}&\int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon}\left(\Delta_\xi u_{j,\varepsilon} - \Delta_{g_{j,\varepsilon}} u_{j,\varepsilon}\right)\,dx\\
&= o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right) + O\left(r_{j,\varepsilon}^3 \int_{B_0(\delta)} |x|^2 \left|\nabla\widetilde{\psi}_{j,\varepsilon}\right|^2\,dx\right)\\
&\quad + O\left(r_{j,\varepsilon}^2 \int_{B_0(\delta)} |x| \left|\nabla\widetilde{\psi}_{j,\varepsilon}\right| |\nabla w_{j,\varepsilon}|\,dx\right) + O\left(r_{j,\varepsilon}^2 \|\nabla w_{j,\varepsilon}\|_2^2\right)\\
&\quad + O\left(r_{j,\varepsilon}^2 \sum_{k\in\mathcal{D}_j} \int_{\Omega_{j,k,\varepsilon}} |x| \left|\nabla\widetilde{\psi}_{j,\varepsilon}\right|^2\,dx\right).\end{aligned}$$

Using (2.14), (2.57) and (2.69), one easily checks that this leads to

$$\int_{\Omega_{j,\varepsilon}(\delta)} \partial_\gamma u_{j,\varepsilon}\left(\Delta_\xi u_{j,\varepsilon} - \Delta_{g_{j,\varepsilon}} u_{j,\varepsilon}\right)\,dx = o\left(\left(\frac{\nu_{j,\varepsilon}}{r_{j,\varepsilon}}\right)^{n-2}\right).$$

Coming back to (2.73) with (2.74), (2.75) and this last relation, we get that $\partial_\gamma h_j(0) = 0$ for all $k = 1, \dots, n$. Thus point (iii) of Claim 2 is proved. q.e.d.

3. Almost isolated concentration points

We consider in this section the case of an almost isolated concentration point. We just sketch the arguments since they mainly follow the lines of those developed in Section 2. We thus refer the reader to the corresponding parts of the previous section for details on some of the assertions below. We let $j \in \{1, \dots, N\}$ and we assume that

$$\lim_{\varepsilon\to 0} r_{j,\varepsilon} = r_0 > 0.$$

Thanks to the definition (2.2) of $r_{j,\varepsilon}$, this implies that

$$u_0 \equiv 0, \tag{3.1}$$

that

$$\mu_{i,\varepsilon} = O\left(\mu_{j,\varepsilon}\right) \text{ for all } i \in \{1, \dots, N\} \tag{3.2}$$

and that

$$\text{for any } i \in \{1, \ldots, N\}, \; i \neq j, \;\; \mu_{j,\varepsilon} = O(\mu_{i,\varepsilon}) \Rightarrow \overline{x}_i \neq \overline{x}_j. \tag{3.3}$$

It comes then from (1.7) that

$$\lim_{\varepsilon \to 0} \mu_{j,\varepsilon}^{1-\frac{n}{2}} u_\varepsilon = a_n^{-\frac{n-2}{2}} (n-2)\, \omega_{n-1} \sum_{i=1}^{N} \left(\lim_{\varepsilon \to 0} \frac{\mu_{i,\varepsilon}}{\mu_{j,\varepsilon}} \right)^{\frac{n}{2}-1} G_0(\overline{x}_i, \, .) \tag{3.4}$$

in $C^2_{\text{loc}}(M \backslash \mathcal{S})$, $\mathcal{S}$ as in (1.2), where G_0 is the Green function of $\Delta_g + h_0$. Let us set

$$\mathcal{C}_j = \left\{ k \in \{1, \ldots, N\}, \; k \neq j \;\text{ s.t. }\; \overline{x}_j = \overline{x}_k \right\}. \tag{3.5}$$

Thanks to (3.2) and (3.3), we have that

$$\text{for any } k \in \mathcal{C}_j, \;\; \mu_{k,\varepsilon} = o(\mu_{j,\varepsilon}). \tag{3.6}$$

We let now

$$s_{j,k,\varepsilon}^2 = \frac{\mu_{k,\varepsilon}}{\mu_{j,\varepsilon}} d_g(x_{k,\varepsilon}, x_{j,\varepsilon})^2 + \mu_{j,\varepsilon}\mu_{k,\varepsilon} \tag{3.7}$$

for $k \in \mathcal{C}_j$. Note that, thanks to (3.6),

$$s_{j,k,\varepsilon} = o(1) \;\text{ for all } k \in \mathcal{C}_j. \tag{3.8}$$

We let now $\mathcal{D}_j$ be a subset of $\mathcal{C}_j$ and $(R_k)_{k \in \mathcal{D}_j}$ be a sequence of positive real numbers such that

$$\text{for any } k, k' \in \mathcal{D}_j, \; k \neq k', \; \frac{d_g\left(x_{k,\varepsilon}, x_{k',\varepsilon}\right)}{s_{j,k,\varepsilon}} \to +\infty \;\text{ as } \varepsilon \to 0 \tag{3.9}$$

and such that

$$\begin{gathered} \text{for any } k' \in \mathcal{C}_j, \; \exists \;\text{ a unique } k \in \mathcal{D}_j \;\text{ such that} \\ \limsup_{\varepsilon \to 0} \frac{d_g\left(x_{k,\varepsilon}, x_{k',\varepsilon}\right)}{s_{j,k,\varepsilon}} \leq \frac{R_k}{10} \;\text{ and }\; \limsup_{\varepsilon \to 0} \frac{s_{j,k',\varepsilon}}{s_{j,k,\varepsilon}} \leq \frac{R_k}{10}. \end{gathered} \tag{3.10}$$

We claim that there exists $C > 0$ independent of ε such that for any $k \in \mathcal{D}_j$,

$$\begin{gathered} \text{for any } x \in B_{x_{k,\varepsilon}}(R_k s_{j,k,\varepsilon}) \backslash B_{x_{k,\varepsilon}}\left(\frac{R_k}{4} s_{j,k,\varepsilon} \right), \\ |\nabla u_\varepsilon|_g(x) \leq C \mu_{k,\varepsilon}^{\frac{n}{2}-1} s_{j,k,\varepsilon}^{1-n}, \;\; u_\varepsilon(x) \leq C \mu_{k,\varepsilon}^{\frac{n}{2}-1} s_{j,k,\varepsilon}^{2-n}. \end{gathered} \tag{3.11}$$

The proof of such a claim is based on Claim 1 and follows exactly the proof of (2.14) in Section 2. We let $\eta : [0,+\infty[\mapsto \mathbb{R}$ be a smooth function verifying that $\eta \equiv 1$ on $\left[0,\frac{1}{4}\right]$ and $\eta \equiv 0$ on $\left[\frac{1}{2},+\infty\right[$. We set

$$\sigma_{j,\varepsilon} = \prod_{k\in\mathcal{D}_j}\left(1-\eta\left(\frac{d_g\left(x_{k,\varepsilon},\,.\,\right)}{R_k s_{j,k,\varepsilon}}\right)\right) \text{ and } v_{j,\varepsilon} = \sigma_{j,\varepsilon}u_\varepsilon. \tag{3.12}$$

It is easily checked thanks to (1.3) that there exists $C>0$ independent of ε such that

$$v_{j,\varepsilon} \le C\varphi_{j,\varepsilon} \text{ in } B_{x_{j,\varepsilon}}\left(\delta_j\right) \tag{3.13}$$

where $\delta_j \in \mathbb{R}_+^\star$ is fixed such that

$$\delta_j \le \frac{1}{2}\min\left\{d_g\left(\overline{x}_j,\overline{x}_k\right),\, k\in\{1,\dots,N\}\setminus\mathcal{C}_j,\, k\neq j\right\} \text{ and } \delta_j \le \frac{1}{6}i_g(M).$$

We set

$$\Lambda_{j,\varepsilon} = \Big\{ (y,\nu,\theta)\in M\times\mathbb{R}_+^\star\times\mathbb{R} \text{ s.t. } d_g\left(x_{j,\varepsilon},y\right)\le\mu_{j,\varepsilon}\,,\ \frac{1}{2}\le\frac{\nu}{\mu_{j,\varepsilon}}\le 2\,,\ -\frac{1}{2}\le\theta\le\frac{1}{2}\Big\} \tag{3.14}$$

and we let $(y_{j,\varepsilon},\nu_{j,\varepsilon},\theta_{j,\varepsilon})\in\Lambda_{j,\varepsilon}$ be such that

$$J_{j,\varepsilon}\big(y_{j,\varepsilon},\nu_{j,\varepsilon},\theta_{j,\varepsilon}\big) = \min_{(y,\nu,\theta)\in\Lambda_{j,\varepsilon}} J_{j,\varepsilon}\big(y,\nu,\theta\big) \tag{3.15}$$

where

$$J_{j,\varepsilon}\big(y,\nu,\theta\big) = \int_M \left|\nabla\left(\eta\left(\frac{d_g\left(y,\,.\,\right)}{2\delta_j}\right)\left(v_{j,\varepsilon}-(1+\theta)\,\psi_{y,\nu}\right)\right)\right|_g^2 dv_g$$

with $\psi_{y,\nu}$ as in (2.23). Let us prove that

$$J_{j,\varepsilon}\left(y_{j,\varepsilon},\nu_{j,\varepsilon},\theta_{j,\varepsilon}\right)\to 0 \text{ as } \varepsilon\to 0. \tag{3.16}$$

Note that $(x_{j,\varepsilon},\mu_{j,\varepsilon},0)\in\Lambda_{j,\varepsilon}$ so that

$$J_{j,\varepsilon}\left(y_{j,\varepsilon},\nu_{j,\varepsilon},\theta_{j,\varepsilon}\right)\le J_{j,\varepsilon}\left(x_{j,\varepsilon},\mu_{j,\varepsilon},0\right).$$

We write then that

$$\begin{aligned} J_{j,\varepsilon}\left(x_{j,\varepsilon},\mu_{j,\varepsilon},0\right) &= \int_M \left|\nabla\left(\eta\left(\frac{d_g\left(x_{j,\varepsilon},.\right)}{2\delta_j}\right)\left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)\right)\right|_g^2 dv_g \\ &\le C\int_{B_{x_{j,\varepsilon}}(\delta_j)\backslash B_{x_{j,\varepsilon}}\left(\frac{\delta_j}{2}\right)} \left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)^2 dv_g \\ &\quad +C\int_{B_{x_{j,\varepsilon}}(\delta_j)} \left|\nabla\left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)\right|_g^2 dv_g \end{aligned}$$

where $C>0$ is some constant independent of ε. Thanks to (3.13), it is easily checked that

$$\lim_{\varepsilon\to 0}\int_{B_{x_{j,\varepsilon}}(\delta_j)\backslash B_{x_{j,\varepsilon}}\left(\frac{\delta_j}{2}\right)} \left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)^2 dv_g = 0$$

so that we obtain that

$$J_{j,\varepsilon}\left(y_{j,\varepsilon},\nu_{j,\varepsilon},\theta_{j,\varepsilon}\right) = O\left(\int_{B_{x_{j,\varepsilon}}(\delta_j)} \left|\nabla\left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)\right|_g^2 dv_g\right). \tag{3.17}$$

We write with (1.5) and (3.1) that

$$\begin{aligned} &\int_{B_{x_{j,\varepsilon}}(\delta_j)} \left|\nabla\left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)\right|_g^2 dv_g \\ &\le C\int_{B_{x_{j,\varepsilon}}(\delta_j)} \left|\nabla\left(\left(\sigma_{j,\varepsilon}-1\right)\varphi_{j,\varepsilon}\right)\right|_g^2 dv_g \\ &\quad +C\sum_{i\neq j}\int_{B_{x_{j,\varepsilon}}(\delta_j)} \left|\nabla\left(\sigma_{j,\varepsilon}\varphi_{i,\varepsilon}\right)\right|_g^2 dv_g \\ &\quad +C\int_{B_{x_{j,\varepsilon}}(\delta_j)} \left|\nabla\left(\sigma_{j,\varepsilon}R_\varepsilon\right)\right|_g^2 dv_g. \end{aligned}$$

with $\|R_\varepsilon\|_{H_1^2(M)}\to 0$ as $\varepsilon\to 0$. Following the proofs of (2.27) – (2.29) and of the relation preceding (2.30), we then estimate all the terms of the right-hand side of this relation. This leads to

$$\lim_{\varepsilon\to 0}\int_{B_{x_{j,\varepsilon}}(\delta_j)} \left|\nabla\left(v_{j,\varepsilon}-\varphi_{j,\varepsilon}\right)\right|_g^2 dv_g = 0. \tag{3.18}$$

Thanks to (3.17), this proves (3.16). As in Section 2, we deduce from (3.16) that

$$\lim_{\varepsilon\to 0}\theta_{j,\varepsilon}=0,\ \ \lim_{\varepsilon\to 0}\frac{\mu_{j,\varepsilon}}{\nu_{j,\varepsilon}}=1\ \text{ and }\ \lim_{\varepsilon\to 0}\frac{d_g\left(x_{j,\varepsilon},y_{j,\varepsilon}\right)}{\mu_{j,\varepsilon}}=0. \tag{3.19}$$

Let $0 < 3\delta_j < \delta < \frac{i_g(M)}{2}$. We set for $x \in B_0(\delta)$, the Euclidean ball of center 0 and radius δ,

$$\begin{aligned} g_{j,\varepsilon}(x) &= \exp^{\star}_{y_{j,\varepsilon}} g(x), \\ u_{j,\varepsilon}(x) &= u_\varepsilon\left(\exp_{y_{j,\varepsilon}}(x)\right), \\ h_{j,\varepsilon}(x) &= h_\varepsilon\left(\exp_{y_{j,\varepsilon}}(x)\right), \\ \widetilde{\sigma}_{j,\varepsilon}(x) &= \sigma_{j,\varepsilon}\left(\exp_{y_{j,\varepsilon}}(x)\right) \text{ and} \\ \widetilde{v}_{j,\varepsilon}(x) &= \widetilde{\sigma}_{j,\varepsilon}(x)\, u_{j,\varepsilon}(x) = v_{j,\varepsilon}\left(\exp_{y_{j,\varepsilon}}(x)\right). \end{aligned} \tag{3.20}$$

Note that $g_{j,\varepsilon}$ is controled on both sides by the Euclidean metric in the sense of bilinear forms. Since u_ε verifies Equation (E_ε), $u_{j,\varepsilon}$ verifies

$$\Delta_{g_{j,\varepsilon}} u_{j,\varepsilon} + h_{j,\varepsilon} u_{j,\varepsilon} = u_{j,\varepsilon}^{2^\star - 1} \tag{3.21}$$

in $B_0(\delta)$. Note also that (3.13) just becomes

$$\widetilde{v}_{j,\varepsilon}(x) \le C \widetilde{\psi}_{j,\varepsilon} \text{ in } B_0(\delta_j) \tag{3.22}$$

where

$$\widetilde{\psi}_{j,\varepsilon}(x) = \left(\frac{\nu_{j,\varepsilon}}{\nu_{j,\varepsilon}^2 + a_n |x|^2} \right)^{\frac{n-2}{2}}.$$

We write

$$\eta_j \widetilde{v}_{j,\varepsilon} = (1 + \theta_{j,\varepsilon})\, \eta_j \widetilde{\psi}_{j,\varepsilon} + w_{j,\varepsilon} \tag{3.23}$$

where $w_{j,\varepsilon} \in C_c^\infty\big(B_0(\delta_j)\big)$ and

$$\eta_j = \eta\left(\frac{\cdot}{2\delta_j}\right).$$

We express (3.15). Differentiating $J_{j,\varepsilon}$ with respect to θ, we obtain that

$$\int_{B_0(\delta_j)} \left(\nabla\left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right), \nabla w_{j,\varepsilon}\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} = 0. \tag{3.24}$$

Differentiating $J_{j,\varepsilon}$ with respect to y, we get that

$$\int_{B_0(\delta_j)} \left(\nabla\left(\eta_j \frac{\partial \widetilde{\psi}_{j,\varepsilon}}{\partial x_i}\right), \nabla w_{j,\varepsilon}\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} = O\left(\nu_{j,\varepsilon}^{n-2}\right) \tag{3.25}$$

for all $i = 1, \dots, n$. At last, differentiating $J_{j,\varepsilon}$ with respect to ν, we obtain thanks to (3.24) that

$$\text{(3.26)} \quad \int_{B_0(\delta_j)} \left(\nabla \left(\eta_j |x|^2 \left(1 + a_n \frac{|x|^2}{\nu_{j,\varepsilon}^2} \right)^{-\frac{n}{2}} \right), \nabla w_{j,\varepsilon} \right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} = 0.$$

The aim is to estimate $\int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|^2_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}}$. We write first thanks to (3.23) and (3.24) that

$$\begin{aligned} \int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|^2_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} &= \int_{B_0(\delta_j)} \left(\nabla w_{j,\varepsilon}, \nabla \left(\eta_j \widetilde{v}_{j,\varepsilon} \right) \right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} \\ &= \int_{B_0(\delta_j)} w_{j,\varepsilon} \Delta_{g_{j,\varepsilon}} \left(\eta_j \widetilde{v}_{j,\varepsilon} \right) dv_{g_{j,\varepsilon}}. \end{aligned}$$

Writing thanks to (3.4), (3.19), (3.20), (3.21) and (3.23) that

$$\begin{aligned} w_{j,\varepsilon} \Delta_{g_{j,\varepsilon}} \left(\eta_j \widetilde{v}_{j,\varepsilon} \right) &= \left(\eta_j \widetilde{v}_{j,\varepsilon} \right)^{2^\star - 1} w_{j,\varepsilon} - h_{j,\varepsilon} \left(\eta_j \widetilde{v}_{j,\varepsilon} \right) w_{j,\varepsilon} \\ &\quad + O\left(|w_{j,\varepsilon}| \, |\nabla \widetilde{\sigma}_{j,\varepsilon}|_{g_{j,\varepsilon}} \, |\nabla u_{j,\varepsilon}|_{g_{j,\varepsilon}} \right) \\ &\quad + O\left(|w_{j,\varepsilon}| \left| \Delta_{g_{j,\varepsilon}} \widetilde{\sigma}_{j,\varepsilon} \right| u_{j,\varepsilon} \right) + O\left(\nu_{j,\varepsilon}^{n-2} \right) \\ &\quad + O\left(\left| \widetilde{\sigma}_{j,\varepsilon} - \widetilde{\sigma}_{j,\varepsilon}^{2^\star - 1} \right| u_{j,\varepsilon}^{2^\star - 1} |w_{j,\varepsilon}| \right) \end{aligned}$$

in $B_0(\delta_j)$, we get following the proof of (2.43) and using (0.1), (3.6), (3.7), (3.9), (3.11), (3.12) and (3.22) that

$$\begin{aligned} \text{(3.27)} \quad & \left(1 + o(1) \right) \int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|^2_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} \\ &= \int_{B_0(\delta_j)} \left(\eta_j \widetilde{v}_{j,\varepsilon} \right)^{2^\star - 1} w_{j,\varepsilon} \, dv_{g_{j,\varepsilon}} + o\left(\nu_{j,\varepsilon}^3 \right) + O\left(\nu_{j,\varepsilon}^{\frac{n}{2} - 1} \|\nabla w_{j,\varepsilon}\|_2 \right) \\ &\quad + O\left(\nu_{j,\varepsilon}^{n-2} \right) + O\left(\left(\sum_{k \in \mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}} \right)^{\frac{n}{2} - 1} \right) \|\nabla w_{j,\varepsilon}\|_2 \right). \end{aligned}$$

Relations (3.16) and (3.23) give that $\int_{B_0(\delta_j)} |\nabla w_{j,\varepsilon}|^2_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} \to 0$ as $\varepsilon \to 0$ so that Hölder's and Sobolev's inequalities permit to write with

(3.23) that

$$\int_{B_0(\delta_j)} \left(\eta_j \widetilde{v}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} \tag{3.28}$$
$$= \left(1+\theta_{j,\varepsilon}\right)^{2^\star-1} \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}}$$
$$+ \left(2^\star-1\right)\left(1+\theta_{j,\varepsilon}\right)^{2^\star-2} \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2} w_{j,\varepsilon}^2\, dv_{g_{j,\varepsilon}}$$
$$+ o\left(\int_{B_0(\delta_j)} \left|\nabla w_{j,\varepsilon}\right|_{g_{j,\varepsilon}}^2 dv_{g_{j,\varepsilon}}\right).$$

Direct computations give then thanks to (3.4) and to the Cartan expansion of the metric $g_{j,\varepsilon}$ around 0 that

$$\int_{B_0(\delta_j)} \left(\nabla w_{j,\varepsilon}, \nabla\left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)\right)_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}}$$
$$= \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} + O\left(\int_{B_0(\delta_j)} |x|^2 \widetilde{\psi}_{j,\varepsilon}^{2^\star-1} \left|w_{j,\varepsilon}\right|\, dv_{g_{j,\varepsilon}}\right)$$
$$+ O\left(\int_{B_0(\delta_j)} |x|^2 \left|\nabla \widetilde{\psi}_{j,\varepsilon}\right| \left|\nabla w_{j,\varepsilon}\right|\, dv_{g_{j,\varepsilon}}\right) + O\left(\nu_{j,\varepsilon}^{n-2}\right)$$

so that, using (3.24), Hölder's and Sobolev's inequalities, we get after simple computations that

$$\int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-1} w_{j,\varepsilon}\, dv_{g_{j,\varepsilon}} = O\left(\nu_{j,\varepsilon}^{n-2}\right) + o\left(\nu_{j,\varepsilon}^3\right) + o\left(\left\|\nabla w_{j,\varepsilon}\right\|_2^2\right)$$
$$+ O\left(\nu_{j,\varepsilon}^{\frac{n}{2}-1} \left\|\nabla w_{j,\varepsilon}\right\|_2\right).$$

Coming back to (3.27) with (3.28) and this last estimate, we arrive thanks to (3.19) to

$$\left(1+o(1)\right) \int_{B_0(\delta_j)} \left|\nabla w_{j,\varepsilon}\right|_{g_{j,\varepsilon}}^2 dv_{g_{j,\varepsilon}}$$
$$= \left(2^\star-1\right) \int_{B_0(\delta_j)} \left(\eta_j \widetilde{\psi}_{j,\varepsilon}\right)^{2^\star-2} w_{j,\varepsilon}^2\, dv_{g_{j,\varepsilon}} + O\left(\nu_{j,\varepsilon}^{n-2}\right)$$
$$+ O\left(\left(\sum_{k\in\mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{\frac{n}{2}-1}\right) \left\|\nabla w_{j,\varepsilon}\right\|_2\right) + o\left(\nu_{j,\varepsilon}^3\right) + O\left(\nu_{j,\varepsilon}^{\frac{n}{2}-1} \left\|\nabla w_{j,\varepsilon}\right\|_2\right).$$

Considering an eigenvalue problem like in (2.47), we then follow the lines of Section 2 and use the lemma of Appendix 1 to obtain that

$$\|\nabla w_{j,\varepsilon}\|_2^2 = O\left(\nu_{j,\varepsilon}^{n-2}\right) + o\left(\nu_{j,\varepsilon}^3\right) + O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right). \tag{3.29}$$

We also get, following the proof of (2.58), that

$$\theta_{j,\varepsilon} = O\left(\nu_{j,\varepsilon}^{n-2}\right) + o\big(\nu_{j,\varepsilon}\big) + O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right). \tag{3.30}$$

We now apply the Pohozaev identity with test function $f = \frac{1}{2}|x|^2$, to $u_{j,\varepsilon}$ (see Appendix 2) in

$$\Omega_{j,\varepsilon}(\delta) = B_0(\delta) \setminus \bigcup_{k\in\mathcal{D}_j} \Omega_{j,k,\varepsilon}$$

for some $\delta > 0$ small enough with

$$\Omega_{j,k,\varepsilon} = \exp_{y_{j,\varepsilon}}^{-1}\left(B_{x_{k,\varepsilon}}\left(\frac{R_k}{2}s_{j,k,\varepsilon}\right)\right).$$

We thus have that

$$\begin{aligned}\int_{\Omega_{j,\varepsilon}(\delta)}\left(h_{j,\varepsilon} + \frac{1}{2}\big(\nabla h_{j,\varepsilon}, \nabla f\big)_{g_{j,\varepsilon}} + \frac{1}{4}\left(\Delta_{g_{j,\varepsilon}}\right)^2 f\right) u_{j,\varepsilon}^2\, dv_{g_{j,\varepsilon}} \\ = \frac{1}{n}\int_{\Omega_{j,\varepsilon}(\delta)}\big(\Delta_{g_{j,\varepsilon}} f + n\big) u_{j,\varepsilon}^{2^\star}\, dv_{g_{j,\varepsilon}} \\ + \int_{\Omega_{j,\varepsilon}(\delta)}\big(\nabla^2 f - g_{j,\varepsilon}\big)\big(\nabla u_{j,\varepsilon}, \nabla u_{j,\varepsilon}\big)\, dv_{g_{j,\varepsilon}} + A_\varepsilon\end{aligned} \tag{3.31}$$

where A_ε is the boundary term

$$\begin{aligned}A_\varepsilon = \int_{\partial\Omega_{j,\varepsilon}(\delta)}\Bigg(\frac{1}{2}\left|\nabla u_{j,\varepsilon}\right|_{g_{j,\varepsilon}}^2 \big(\nabla f, \nu\big)_{g_{j,\varepsilon}} \\ - \big(\nabla u_{j,\varepsilon}, \nabla f\big)_{g_{j,\varepsilon}}\big(\nabla u_{j,\varepsilon}, \nu\big)_{g_{j,\varepsilon}}\Bigg)\, d\sigma_{g_{j,\varepsilon}} \\ - \frac{n-2}{2}\int_{\partial\Omega_{j,\varepsilon}(\delta)} u_{j,\varepsilon}\big(\nabla u_{j,\varepsilon}, \nu\big)_{g_{j,\varepsilon}}\, d\sigma_{g_{j,\varepsilon}}\end{aligned}$$

$$
\begin{aligned}
&- \int_{\partial\Omega_{j,\varepsilon}(\delta)} (\nabla f, \nu)_{g_{j,\varepsilon}} \left(\frac{1}{2^\star} u_{j,\varepsilon}^{2^\star} - \frac{1}{2} h_{j,\varepsilon} u_{j,\varepsilon}^2 \right) d\sigma_{g_{j,\varepsilon}} \\
&+ \frac{1}{2} \int_{\partial\Omega_{j,\varepsilon}(\delta)} \left(\Delta_{g_{j,\varepsilon}} f + n \right) u_{j,\varepsilon} \left(\nabla u_{j,\varepsilon}, \nu \right)_{g_{j,\varepsilon}} d\sigma_{g_{j,\varepsilon}} \\
&- \frac{1}{4} \int_{\partial\Omega_{j,\varepsilon}(\delta)} \left(\nabla \left(\Delta_{g_{j,\varepsilon}} f \right), \nu \right)_{g_{j,\varepsilon}} u_{j,\varepsilon}^2 \, d\sigma_{g_{j,\varepsilon}}
\end{aligned}
$$

where ν denotes the unit outer normal of $\partial\Omega_{j,\varepsilon}(\delta)$. Note that, by (3.8) and (3.9),

$$
\partial\Omega_{j,\varepsilon}(\delta) = \cup_{k \in \mathcal{D}_j} \partial\Omega_{j,k,\varepsilon} \cup \partial B_0(\delta)
$$

and the union is a disjoint one. We denote by $A_{k,\varepsilon}$ the part of A_ε corresponding to $\partial\Omega_{j,k,\varepsilon}$. Noting that, on $\partial\Omega_{j,k,\varepsilon}$,

$$
|\nabla f| \le d_g(y_{j,\varepsilon}, x_{k,\varepsilon}) + \frac{R_k}{2} s_{j,k,\varepsilon},
$$

we get thanks to (0.1), (3.6), (3.7), (3.8) and (3.11) that

$$
A_{k,\varepsilon} = O\left(\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}} \right)^{n-2} \left(1 + \frac{d_g(y_{j,\varepsilon}, x_{k,\varepsilon})}{s_{j,k,\varepsilon}} \right) \right)
$$

for all $k \in \mathcal{D}_j$. In order to estimate the part of A_ε corresponding to $\partial B_0(\delta)$, we use (3.4). We finally obtain that

(3.32)

$$
A_\varepsilon = \big(A_j(\delta) + o(1) \big) \nu_{j,\varepsilon}^{n-2} + O\left(\sum_{k \in \mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}} \right)^{n-2} \left(1 + \frac{d_g(x_{k,\varepsilon}, y_{j,\varepsilon})}{s_{j,k,\varepsilon}} \right) \right)
$$

where

(3.33)

$$
\begin{aligned}
A_j(\delta) = &\int_{\partial B_{\overline{x}_j}(\delta)} \left(\frac{1}{2} |\nabla H_j|_g^2 (\nabla f_j, \nu)_g - (\nabla H_j, \nabla f_j)_g (\nabla H_j, \nu)_g \right) d\sigma_g \\
&- \frac{n-2}{2} \int_{\partial B_{\overline{x}_j}(\delta)} H_j (\nabla H_j, \nu)_g \, d\sigma_g \\
&+ \frac{1}{2} \int_{\partial B_{\overline{x}_j}(\delta)} (\nabla f_j, \nu)_g h_0 H_j^2 \, d\sigma_g \\
&+ \frac{1}{2} \int_{\partial B_{\overline{x}_j}(\delta)} \left(\Delta_g f_j + n \right) H_j (\nabla H_j, \nu)_g \, d\sigma_g \\
&- \frac{1}{4} \int_{\partial B_{\overline{x}_j}(\delta)} \left(\nabla (\Delta_g f_j), \nu \right)_g H_j^2 \, d\sigma_g
\end{aligned}
$$

with $f_j = \frac{1}{2} d_g(\overline{x}_j, .)^2$ and

$$H_j(x) = a_n^{-\frac{n-2}{2}} (n-2)\,\omega_{n-1} \sum_{i=1}^{N} \left(\lim_{\varepsilon \to 0} \frac{\mu_{i,\varepsilon}}{\mu_{j,\varepsilon}} \right)^{\frac{n}{2}-1} G_0(\overline{x}_i, .)\,. \tag{3.34}$$

Since $(\widetilde{\psi}_{j,\varepsilon})$ is radially summetrical, we have that

$$\left(\nabla^2 f - g_{j,\varepsilon}\right)\left(\nabla\left(\widetilde{\psi}_{j,\varepsilon}\right), \nabla\left(\widetilde{\psi}_{j,\varepsilon}\right)\right) = 0$$

so that, noting that $\widetilde{\sigma}_{j,\varepsilon} = 1$ and $\eta_j = 1$ in $\Omega_{j,\varepsilon}(\delta)$ for δ small enough, we can write with (3.4), (3.19), (3.23) and the Cartan expansion of the metric $g_{j,\varepsilon}$ around 0 that

$$\begin{aligned} &\int_{\Omega_{j,\varepsilon}(\delta)} \left(\nabla^2 f - g_{j,\varepsilon}\right)\left(\nabla u_{j,\varepsilon}, \nabla u_{j,\varepsilon}\right) dv_{g_{j,\varepsilon}} \\ &= O\left(\int_{\Omega_{j,\varepsilon}(\delta)} |x|^2 \left|\nabla \widetilde{\psi}_{j,\varepsilon}\right|_{g_{j,\varepsilon}} \left|\nabla w_{j,\varepsilon}\right|_{g_{j,\varepsilon}} dv_{g_{j,\varepsilon}} \right) + O\left(\delta^2 \|\nabla w_{j,\varepsilon}\|_2^2\right) \end{aligned}$$

where $O(X)$ means $|O(X)| \le CX$ with C independent of ε and δ. This leads by direct computations thanks to (3.29) to

$$\begin{aligned} &\int_{\Omega_{j,\varepsilon}(\delta)} \left(\nabla^2 f - g_{j,\varepsilon}\right)\left(\nabla u_{j,\varepsilon}, \nabla u_{j,\varepsilon}\right) dv_{g_{j,\varepsilon}} \\ &\quad = O\left(\delta^{\frac{1}{2}} \nu_{j,\varepsilon}^{n-2}\right) + o\left(\nu_{j,\varepsilon}^3\right) + O\left(\delta^{\frac{1}{2}} \sum_{k \in \mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}} \right)^{n-2} \right). \end{aligned} \tag{3.35}$$

The Cartan expansion of the metric $g_{j,\varepsilon}$ around 0 gives that

$$\begin{aligned} \Delta_{g_{j,\varepsilon}} f + n = {}& \frac{1}{3} \mathrm{Ric}_{g_{j,\varepsilon}}(0)_{kl}\, x^k x^l \\ & + \frac{1}{2} \partial_{klm} \left(\ln \sqrt{|g_{j,\varepsilon}|} \right)(0) x^k x^l x^m + O\left(|x|^4\right) \end{aligned}$$

in $B_0(\delta)$ where $\mathrm{Ric}_{g_{j,\varepsilon}}$ denotes the Ricci curvature of $g_{j,\varepsilon}$ and where $|g_{j,\varepsilon}|$ is the determinant of the matrix $\left((g_{j,\varepsilon})_{ik}\right)$. Thus we can write thanks to (3.4), (3.19), (3.23) and to the fact that $\left(\widetilde{\psi}_{j,\varepsilon}\right)$ is radially symmetrical that

$$\begin{aligned}&\int_{\Omega_{j,\varepsilon}(\delta)}\left(\Delta_{g_{j,\varepsilon}}f+n\right)u_{j,\varepsilon}^{2^\star}\,dv_{g_{j,\varepsilon}}\\&=\frac{\left(1+\theta_{j,\varepsilon}\right)^{2^\star}}{3}\mathrm{Ric}_{g_{j,\varepsilon}}(0)_{kl}\int_{B_0(\delta)}x^kx^l\widetilde{\psi}_{j,\varepsilon}^{2^\star}\,dx\\&\quad+O\left(\sum_{k\in\mathcal{D}_j}\int_{\Omega_{j,k,\varepsilon}}|x|^2\widetilde{\psi}_{j,\varepsilon}^{2^\star}\,dx\right)+O\left(\|w_{j,\varepsilon}\|_{2^\star}^{2^\star}\right)+O\left(\nu_{j,\varepsilon}^n\right)\\&\quad+O\left(\int_{B_0(\delta)}|x|^2\widetilde{\psi}_{j,\varepsilon}^{2^\star-1}|w_{j,\varepsilon}|\,dx\right)+O\left(\int_{B_0(\delta)}|x|^4\widetilde{\psi}_{j,\varepsilon}^{2^\star}\,dx\right)\,.\end{aligned}$$

Using (3.29) and (3.30), this leads thanks to Hölder's and Sobolev's inequalities and to some explicit computations to the following:

$$\begin{aligned}\text{(3.36)}\quad\int_{\Omega_{j,\varepsilon}}\left(\Delta_{g_{j,\varepsilon}}f+n\right)u_{j,\varepsilon}^{2^\star}\,dv_{g_{j,\varepsilon}}&=\frac{n}{3}K_n^{-\frac{n}{2}}S_g\left(y_{j,\varepsilon}\right)\nu_{j,\varepsilon}^2+o\left(\nu_{j,\varepsilon}^{n-2}\right)\\&\quad+o\left(\nu_{j,\varepsilon}^3\right)+O\left(\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right).\end{aligned}$$

Let us estimate the last term of (3.31). Using (0.1) and the Cartan expansion of the metric $g_{j,\varepsilon}$ around 0, one gets that

$$\begin{aligned}&h_{j,\varepsilon}+\frac{1}{2}\left(\nabla h_{j,\varepsilon},\nabla f\right)_{g_{j,\varepsilon}}+\frac{1}{4}\left(\Delta_{g_{j,\varepsilon}}\right)^2f\\&=\left(h_\varepsilon\left(y_{j,\varepsilon}\right)-\frac{1}{6}S_g\left(y_{j,\varepsilon}\right)\right)\\&\quad+\left(\frac{3}{2}\partial_kh_{j,\varepsilon}(0)+\frac{1}{4}\partial_k\left(\Delta_{g_{j,\varepsilon}}^2f\right)(0)\right)x^k+O\left(|x|^2\right)\end{aligned}$$

in $B_0(\delta)$ so that we obtain with (3.4), (3.19), (3.23) and the fact that

$\left(\widetilde{\psi}_{j,\varepsilon}\right)$ is radially symmetrical that

$$\begin{aligned}&\int_{\Omega_{j,\varepsilon}(\delta)}\left(h_{j,\varepsilon}+\frac{1}{2}(\nabla h_{j,\varepsilon},\nabla f)_{g_{j,\varepsilon}}+\frac{1}{4}\left(\Delta_{g_{j,\varepsilon}}\right)^2 f\right)u_{j,\varepsilon}^2\,dv_{g_{j,\varepsilon}}\\&=\left(h_\varepsilon\left(y_{j,\varepsilon}\right)-\frac{1}{6}S_g\left(y_{j,\varepsilon}\right)\right)(1+\theta_{j,\varepsilon})^2\int_{B_0(\delta)}\widetilde{\psi}_{j,\varepsilon}^2\,dx\\&\quad+O\left(\sum_{k\in\mathcal{D}_j}\int_{\Omega_{j,k,\varepsilon}}\widetilde{\psi}_{j,\varepsilon}^2\,dx\right)+O\left(\int_{B_0(\delta)}|x|^2\widetilde{\psi}_{j,\varepsilon}^2\,dx\right)\\&\quad+O\left(\int_{B_0(\delta)}\widetilde{\psi}_{j,\varepsilon}|w_{j,\varepsilon}|\,dx\right)+O\left(\int_{B_0(\delta)}|w_{j,\varepsilon}|^2\,dx\right).\end{aligned}$$

This leads thanks to (3.29) and (3.30) and after direct computations to

$$\begin{aligned}&\int_{\Omega_{j,\varepsilon}(\delta)}\left(h_{j,\varepsilon}+\frac{1}{2}(\nabla h_{j,\varepsilon},\nabla f)_{g_{j,\varepsilon}}+\frac{1}{4}\left(\Delta_{g_{j,\varepsilon}}\right)^2 f\right)u_{j,\varepsilon}^2\,dv_{g_{j,\varepsilon}}\\&=\left(h_\varepsilon\left(y_{j,\varepsilon}\right)-\frac{1}{6}S_g\left(y_{j,\varepsilon}\right)\right)\times\begin{cases}64\omega_3\nu_{j,\varepsilon}^2\ln\left(\dfrac{r_{j,\varepsilon}}{\nu_{j,\varepsilon}}\right) & \text{if } n=4\\[2ex] \dfrac{4(n-1)}{n-4}K_n^{-\frac{n}{2}}\nu_{j,\varepsilon}^2 & \text{if } n\geq 5\end{cases}\\&\quad+O\left(\delta^{\frac{1}{2}}\nu_{j,\varepsilon}^{n-2}\right)+o\left(\nu_{j,\varepsilon}^3\right)+O\left(\delta^{\frac{1}{2}}\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\right).\end{aligned}$$

Coming back to (3.31) with (3.32), (3.35), (3.36) and this last estimate, we finally arrive to

(3.37)

$$\int_{B_{\overline{x}_j}(\delta)}\left(h_0+\frac{1}{2}\left(\nabla h_0,\nabla f_j\right)_g+\frac{1}{4}\left(\Delta_g\right)^2 f_j\right)H_j^2\,dv_g=A_j\left(\delta\right)+o(1)$$

with $o\left(1\right)\to 0$ as $\delta\to 0$ when $n=3$. Remember here that, when $n=3$, $\mathcal{D}_j=\emptyset$ thanks to Claim 2. When $n=4$, we get that

(3.38)

$$\begin{aligned}&\left(h_\varepsilon\left(y_{j,\varepsilon}\right)-\frac{1}{6}S_g\left(y_{j,\varepsilon}\right)\right)\ln\left(\frac{1}{\nu_{j,\varepsilon}}\right)\\&=\frac{A_j\left(\delta\right)}{64\omega_3}+O\left(\sqrt{\delta}\right)+O\left(\nu_{j,\varepsilon}^{-2}\sum_{k\in\mathcal{D}_j}\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2}\left(1+\frac{d_g\left(x_{k,\varepsilon},y_{j,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)\right)\end{aligned}$$

and when $n \geq 5$, we arrive to

$$\begin{aligned}(3.39)\qquad &\left(h_\varepsilon\left(y_{j,\varepsilon}\right) - C(n) S_g\left(y_{j,\varepsilon}\right)\right)\\ &= \left(\frac{n-4}{4(n-1)} K_n^{\frac{n}{2}} A_j(\delta) + O\left(\sqrt{\delta}\right)\right) \nu_{j,\varepsilon}^{n-4}\\ &\quad + O\left(\nu_{j,\varepsilon}^{-2} \sum_{k\in\mathcal{D}_j} \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2} \left(1 + \frac{d_g\left(x_{k,\varepsilon}, y_{j,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)\right).\end{aligned}$$

We assume for the end of this section that $n \geq 4$. Let $k \in \mathcal{D}_j$. We write with (3.7) and (3.19) that

$$\frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}} = \frac{d_g\left(x_{j,\varepsilon}, x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}} + o\left(\frac{\mu_{j,\varepsilon}}{s_{j,k,\varepsilon}}\right) = O\left(\sqrt{\frac{\mu_{j,\varepsilon}}{\mu_{k,\varepsilon}}}\right).$$

Thanks to (3.6), $j \in \mathcal{A}_k$ so that, by (2.2), we get that $s_{j,k,\varepsilon} \geq r_{k,\varepsilon}$. In particular, by (3.8), $r_{k,\varepsilon} \to 0$ as $\varepsilon \to 0$ and we can apply Claim 2, point (ii), to get that

$$\begin{aligned}&\left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2} \left(1 + \frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)\\ &\leq \left(\frac{\mu_{k,\varepsilon}}{r_{k,\varepsilon}}\right)^{n-2} \left(1 + \frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right)\\ &= O\left(\sqrt{\frac{\mu_{j,\varepsilon}}{\mu_{k,\varepsilon}}} \left(\frac{\mu_{k,\varepsilon}}{r_{k,\varepsilon}}\right)^{n-2}\right)\\ &= \begin{cases} O\left(\sqrt{\frac{\mu_{j,\varepsilon}}{\mu_{k,\varepsilon}}} \mu_{k,\varepsilon}^2 \ln\left(\frac{1}{\mu_{k,\varepsilon}}\right)\right) & \text{if } n = 4\\ O\left(\sqrt{\frac{\mu_{j,\varepsilon}}{\mu_{k,\varepsilon}}} \mu_{k,\varepsilon}^2\right) & \text{if } n \geq 5.\end{cases}\end{aligned}$$

Thanks to (3.6)and (3.19), this leads to

$$(3.40)\qquad \left(\frac{\mu_{k,\varepsilon}}{s_{j,k,\varepsilon}}\right)^{n-2} \left(1 + \frac{d_g\left(y_{j,\varepsilon}, x_{k,\varepsilon}\right)}{s_{j,k,\varepsilon}}\right) = \begin{cases} o\left(\nu_{j,\varepsilon}^2 \ln\left(\frac{1}{\nu_{j,\varepsilon}}\right)\right) & \text{if } n = 4\\ o\left(\nu_{j,\varepsilon}^2\right) & \text{if } n \geq 5\end{cases}$$

for all $k \in \mathcal{D}_j$.

4. Proof of the results and examples of blow-up

4.1 Proof of the theorem

Let us prove our theorem as stated in the introduction. We assume that $3 \le n \le 5$. If $u_0 \not\equiv 0$, we know by (2.2) that $r_{1,\varepsilon}^2 \le \mu_{1,\varepsilon}$. In particular, $r_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$ and claim 2 holds. Plugging the estimate $r_{1,\varepsilon}^2 \le \mu_{1,\varepsilon}$ into the estimate of Claim 2, point (ii), we get a contradiction for $n = 4, 5$. For $n = 3$, since Claim 2 says that $r_{1,\varepsilon}$ does not go to 0 as $\varepsilon \to 0$, we also get a contradiction. This proves that for $3 \le n \le 5$, $u_0 \equiv 0$. We assume now that $n = 4, 5$. We claim that

$$h_0\left(\overline{x}_N\right) = C(n) S_g\left(\overline{x}_N\right), \tag{4.1}$$

an assertion which clearly implies the second part of the theorem. Let us prove (4.1). We distinguish two cases. The easiest one is when $(x_{N,\varepsilon}, \mu_{N,\varepsilon})$ is an almost isolated concentration point, that is when $\liminf_{\varepsilon\to 0} r_{N,\varepsilon} > 0$. In this case, we can apply the results of Section 3: relations (3.38) and (3.39) together with (3.40) clearly give, since $\nu_{N,\varepsilon} \to 0$ as $\varepsilon \to 0$, that $h_\varepsilon\left(y_{N,\varepsilon}\right) - C(n) S_g\left(y_{N,\varepsilon}\right) \to 0$ as $\varepsilon \to 0$. By (0.1) and (3.19), this proves (4.1) in this case. Let us now consider the case when $r_{N,\varepsilon} \to 0$ as $\varepsilon \to 0$. Applying point (ii) of Claim 2, we first get that $h_0\left(\overline{x}_N\right) \ge C(n) S_g\left(\overline{x}_N\right)$. Assume by contradiction that

$$h_0\left(\overline{x}_N\right) > C(n) S_g\left(\overline{x}_N\right). \tag{4.2}$$

Since $r_{N,\varepsilon} \to 0$ as $\varepsilon \to 0$ and $u_0 \equiv 0$, (2.2) gives the existence of some $i \in \mathcal{A}_N$ such that

$$\frac{\mu_{N,\varepsilon}}{\mu_{i,\varepsilon}} d_g\left(x_{i,\varepsilon}, x_{N,\varepsilon}\right)^2 \to 0 \text{ as } \varepsilon \to 0.$$

By (1.6), $\mu_{N,\varepsilon} \ge \mu_{i,\varepsilon}$ so that $\overline{x}_i = \overline{x}_N$. This proves that

$$\mathcal{C} = \left\{ i \in \mathcal{A}_N \text{ s.t. } \overline{x}_i = \overline{x}_N \right\} \cup \left\{ N \right\}$$

possesses at least two elements. We let $k \in \mathcal{C}$ be such that, after passing to a subsequence,

$$r_{k,\varepsilon} = \min_{i\in\mathcal{C}} r_{i,\varepsilon}$$

and we let

$$\mathcal{D} = \left(\mathcal{A}_k \cap \mathcal{B}_k\right) \cup \left\{ k \right\}$$

with $\mathcal{B}_k$ as in Claim 2. It is easily checked with the above choice of k that for any $i \in \mathcal{D}$, $\mathcal{D} = \left(\mathcal{A}_i \cap \mathcal{B}_i\right) \cup \left\{i\right\}$ and that $\mathcal{D}$ possesses at least two elements. Moreover, it is clear that $\lambda_{ij} > 0$ for all $i, j \in \mathcal{D}$, $i \neq j$, λ_{ij} as in Claim 2. We let now $R_\varepsilon \to 0$ as $\varepsilon \to 0$ be such that $\{x_{i,\varepsilon},\, i \in \mathcal{D}\} \subset B_{x_{k,\varepsilon}}(R_\varepsilon)$ and such that there exists $j \in \mathcal{D}$ such that $x_{j,\varepsilon} \in \partial B_{x_{k,\varepsilon}}(R_\varepsilon)$. It is then easily checked that all the z_{ji}'s of Claim 2, $i \in \mathcal{D}$, lie in an Euclidean ball whose boundary contains 0. This implies that $\nabla h_j(0) \neq 0$ since $\mathcal{A}_j \cap \mathcal{B}_j \neq \emptyset$ and $\lambda_{ji} > 0$ for all $i \in \mathcal{A}_j \cap \mathcal{B}_j$. Assuming that (4.2) holds, point (iii) of Claim 2 gives that $\nabla h_j(0) = 0$. This is a contradiction. Thus (4.2) is false. This proves (4.1) so that, if $n = 4,\, 5$, there exists at least one geometric concentration point $x_0 \in \mathcal{S}$ such that $h_0(x_0) = C(n)S_g(x_0)$. As a consequence, compactness holds, that is (0.4) holds, if $h_0 \neq C(n)S_g$ everywhere in M.

Let us prove the last part of the theorem. We assume that $h_\varepsilon(x) \leq C(n)S_g(x)$ for all $x \in M$ and all $\varepsilon > 0$. In this situation, we clearly get from (2.66) and (2.67) that $\liminf_{\varepsilon\to 0} r_{1,\varepsilon} > 0$ if $n = 4,\, 5$. Remember here that $h_1(0) > 0$ by (2.7) and that $\mathcal{D}_1 = \emptyset$. Note that, by Claim 2, we also have that $\liminf_{\varepsilon\to 0} r_{1,\varepsilon} > 0$ if $n = 3$ without any assumption, except (0.1), on h_ε. Thus, in this situation, for $n = 3,\, 4,\, 5$, $(x_{1,\varepsilon}, \mu_{1,\varepsilon})$ is an isolated concentration point in the sense that $\mathcal{C}_1 = \emptyset$, $\mathcal{C}_1$ as in (3.5). Applying the results of Section 3 and in particular (3.37) – (3.39), we get that

$$\int_{B_{\overline{x}_1}(\delta)} \left(h_0 + \frac{1}{2}\left(\nabla h_0, \nabla f_1\right)_g + \frac{1}{4}\left(\Delta_g\right)^2 f_1\right) H_1^2\, dv_g = A_1(\delta) + o(1)$$

when $n = 3$ and that $A_1(\delta) \leq O(\delta)$ if $n = 4,\, 5$ for all $\delta > 0$. Here, $A_1(\delta)$ is as in (3.33). Moreover, when $n = 4,\, 5$, since $\nu_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$ thanks to (3.19), we also get that $h_0(\overline{x}_1) = C(n)S_g(\overline{x}_1)$. Let us now write that

$$G_0(\overline{x}_1,\, .) = \widetilde{G}_0(\overline{x}_1,\, .) + \alpha_0(\,.\,)$$

where G_0 and $\widetilde{G}_0$ are the Green functions of $\Delta_g + h_0$ and $\Delta_g + C(n)S_g$ respectively. We have that

$$\Delta_g \alpha_0 + h_0\alpha_0 = \left(C(n)S_g - h_0\right)\widetilde{G}_0(\overline{x}_1,\, .)\,.$$

By standard properties of the Green function, we know that

$$\widetilde{G}_0(\overline{x}_1, x) \leq C d_g(\overline{x}_1, x)^{2-n}$$

for some $C > 0$. If $n = 4, 5$, since $h_0 \le C(n)S_g$ and $h_0(\overline{x}_1) = C(n)S_g(\overline{x}_1)$, we also have that

$$|h_0(x) - C(n)S_g(x)| \le C d_g(\overline{x}_1, x)^2$$

for some $C > 0$. This permits to prove that $\Delta_g \alpha_0 + h_0 \alpha_0 \in L^p(M)$ for all $p < n$. By Sobolev's embedding theorem, we then get that $\alpha_0 \in C^0(M)$. Thanks to the maximum principle, we get that $\alpha_0 > 0$ in M except if $h_0 \equiv C(n)S_g$ in which case $\alpha_0 \equiv 0$. Up to change from the beginning g into $\varphi^{\frac{4}{n-2}} g$, u_ε into $u_\varepsilon \varphi^{-1}$ and h_ε into

$$h_\varepsilon := C(n)S_{\varphi^{\frac{4}{n-2}} g} + (h_\varepsilon - C(n)S_g)\, \varphi^{-\frac{4}{n-2}},$$

we may assume without loss of generality that for all the geometric concentration points $\overline{x}_i \in \mathcal{S}$ (which are the same for u_ε and $u_\varepsilon \varphi^{-1}$), the Green function of $\Delta_g + C(n)S_g$ writes as

$$\widetilde{G}_0(\overline{x}_i, x) = \frac{1}{(n-2)\omega_{n-1} d_g(\overline{x}_i, x)^{n-2}} + \widetilde{M}(\overline{x}_i) + o(1)$$

for x close to $\overline{x}_i$. For the existence of a conformal change of metric having this property in dimensions $n = 3, 4, 5$, we refer to Lee-Parker [21]. Moreover, thanks to the positive mass theorem of Schoen-Yau [31, 32] (see also Schoen [28], Schoen-Yau [33] and Witten [36]), we know that $\widetilde{M}(\overline{x}_i) > 0$ except if (M, g) is conformally diffeomorphic to the standard sphere. Since $\alpha_0 \in C^0(M)$ and $\alpha_0 > 0$ except if $h_0 \equiv C(n)S_g$, we get that

$$G_0(\overline{x}_1, x) = \frac{1}{(n-2)\omega_{n-1} d_g(\overline{x}_1, x)^{n-2}} + M(\overline{x}_1) + o(1)$$

for x close to $\overline{x}_1$ with $M(\overline{x}_1) > 0$ except if (M, g) is conformally diffeomorphic to the standard sphere and $h_0 \equiv C(n)S_g$. Coming back to the definition (3.34) of H_1, we also have that

$$H_1(x) = a_n^{-\frac{n-2}{2}} \left(\frac{1}{d_g(\overline{x}_1, x)^{n-2}} + M_1 \right) + o(1)$$

for x close to $\overline{x}_1$ with $M_1 > 0$ except if (M, g) is conformally diffeomorphic to the standard sphere, $h_0 \equiv C(n)S_g$ and there is only one concentration point. Some computations, which can be carried in conformal normal coordinates (see Lee-Parker [21]) up to the above conformal change of metric, lead then to

$$\lim_{\delta\to 0} A_1(\delta) = \frac{(n-2)^2}{2} a_n^{2-n}\omega_{n-1}M_1$$

for $n = 3$, 4, 5 and to

$$\lim_{\delta\to 0}\int_{B_{\overline{x}_1}(\delta)}\left(h_0 + \frac{1}{2}\left(\nabla h_0, \nabla f_1\right)_g + \frac{1}{4}\left(\Delta_g\right)^2 f_1\right) H_1^2\, dv_g = 0$$

if $n = 3$. Thus we get a contradiction when $n = 3$, 4, 5 and $h_0 \le C(n)S_g$ except if (M, g) is conformally diffeomorphic to the standard sphere and $h_0 \equiv C(n)S_g$. This ends the proof of the theorem. q.e.d.

Let us now state some results in higher dimensions. We claim that if $n \ge 7$, then for any $x \in \mathcal{S}$, $h_0(x) = C(n)S_g(x)$. Of course, this implies again that compactness holds, that is that (0.4) holds, if $h_0 \ne C(n)S_g$ everywhere in M and $n \ge 7$. As shown by the examples below (see Section 4.2), the situation in dimension $n = 6$ is more intricate. However, the above statement continues to hold for $n = 6$ if $u_0 \equiv 0$. Let us prove this statement. We assume that $n = 6$ and $u_0 \equiv 0$ or that $n \ge 7$. We let $i \in \{1, \dots, N\}$. We need to prove that $h_0(\overline{x}_i) = C(n)S_g(\overline{x}_i)$. Up to change i, we may assume that $\mu_{i,\varepsilon} \ge \mu_{j,\varepsilon}$ for all $j \in \{1, \dots, N\}$ such that $\overline{x}_i = \overline{x}_j$. If $(x_{i,\varepsilon}, \mu_{i,\varepsilon})$ is an almost isolated concentration point, then it is a direct consequence of (3.38)–(3.40) together with (0.1) and (3.19) that $h_0(\overline{x}_i) = C(n)S_g(\overline{x}_i)$. We assume now that $r_{i,\varepsilon} \to 0$ as $\varepsilon \to 0$. Then point (ii) of Claim 2 implies that $h_0(\overline{x}_i) \ge C(n)S_g(\overline{x}_i)$. Assume by contradiction that

$$h_0(\overline{x}_i) > C(n)S_g(\overline{x}_i)\ . \tag{4.3}$$

We get then by point (ii) of Claim 2 that

$$r_{i,\varepsilon}^{n-2} = O\left(\mu_{i,\varepsilon}^{n-4}\right). \tag{4.4}$$

In particular, we obtain for $n \ge 7$ that $r_{i,\varepsilon} = o\left(\sqrt{\mu_{i,\varepsilon}}\right)$. Thus, by (2.2), there exists $l \in \mathcal{A}_i$ such that

$$r_{i,\varepsilon}^2 = \frac{\mu_{i,\varepsilon}}{\mu_{l,\varepsilon}} d_g\left(x_{l,\varepsilon}, x_{i,\varepsilon}\right)^{n-2} + \mu_{i,\varepsilon}\mu_{l,\varepsilon}.$$

If $\overline{x}_l \neq \overline{x}_i$, we then get that $r_{i,\varepsilon}^2 \geq C\frac{\mu_{i,\varepsilon}}{\mu_{l,\varepsilon}}$ for some $C > 0$, a contradiction with (4.4) for $n \geq 6$ since $\mu_{i,\varepsilon} \to 0$ and $\mu_{l,\varepsilon} \to 0$ as $\varepsilon \to 0$. Thus $r_{i,\varepsilon}$ is achieved for some $l \in \mathcal{A}_i$ with $\overline{x}_i = \overline{x}_l$. This proves that

$$\mathcal{C} = \left\{k \in \mathcal{A}_i \text{ s.t. } \overline{x}_i = \overline{x}_k\right\} \cup \left\{i\right\}.$$

possesses at least two elements. We let $k \in \mathcal{C}$ be such that, after passing to a subsequence,

$$r_{k,\varepsilon} = \min_{i\in\mathcal{C}} r_{i,\varepsilon}$$

and we let

$$\mathcal{D} = \mathcal{A}_k \cup \mathcal{B}_k \cup \left\{k\right\}$$

with $\mathcal{B}_k$ as in Claim 2. Arguing as above in the low-dimensional case, one can conclude that (4.3) is false and thus end the proof of the above statement. Namely: if $n = 6$ and $u_0 \equiv 0$ or if $n \geq 7$, then $h_0(x) = C(n)S_g(x)$ for all $x \in \mathcal{S}$.

As a remark, the compactness result we obtain in the theorem and in dimensions $n \geq 7$ answers a question asked in Hebey [19] (question Q10). In this paper, the author is mainly interested by the energy function, defined as the minimal energy a solution of $\Delta_g u + \alpha u = u^{2^\star-1}$, $\alpha \in \mathbb{R}$, can have. It follows also from the above results that this energy function is lower semi-continuous with respect to α for $\alpha > C(n)\max_M S_g$ in dimensions $n \geq 4$ (see question Q6 of [19]).

4.2 Examples of blowing-up sequences

We provide in this section some blowing up sequences (u_ε) of solutions of (E_ε). We follow Druet-Hebey [11]. We consider (S^n, h) the unit sphere of $\mathbb{R}^{n+1}$ equipped with its round metric. Its scalar curvature is $n(n-1)$ so that $C(n)S_h = \frac{n(n-2)}{4}$. It is well-known that all the solutions of

$$\Delta_h u + C(n)S_h u = u^{2^\star-1}$$

are given by

$$u = \left(\frac{n(n-2)}{4}\left(\beta^2-1\right)\right)^{\frac{n-2}{4}}\left(\beta - \cos\, d_h\left(x_0, x\right)\right)^{1-\frac{n}{2}}$$

where $\beta > 1$ and $x_0 \in S^n$. For $i = 1, 2$, we let $(x_{i,\varepsilon})_{\varepsilon>0}$ be two sequences of points in S^n and $(\beta_{i,\varepsilon})$ be two sequences of positive real numbers such that $\beta_{i,\varepsilon} > 1$. We let in the following

$$B_{i,\varepsilon} = \left(\frac{n(n-2)}{4}\left(\beta_{i,\varepsilon}^2 - 1\right)\right)^{\frac{n-2}{4}} \left(\beta_{i,\varepsilon} - \cos d_h\left(x_{i,\varepsilon}, x\right)\right)^{1-\frac{n}{2}}$$

for $i = 1, 2$ and

$$u_\varepsilon = \lambda B_{1,\varepsilon} + B_{2,\varepsilon}$$

where $\lambda \in \mathbb{R}^+$. Clearly, $u_\varepsilon \in C^\infty(S^n)$ and $u_\varepsilon > 0$ for all $\varepsilon > 0$. It is easily checked that u_ε verifies

$$\Delta_h u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^\star - 1}$$

in S^n where the linear term h_ε is given by

$$h_\varepsilon = C(n) S_h + \Phi_\varepsilon$$

with

$$\Phi_\varepsilon = \frac{\left(\lambda B_{1,\varepsilon} + B_{2,\varepsilon}\right)^{2^\star - 1} - \lambda B_{1,\varepsilon}^{2^\star - 1} - B_{2,\varepsilon}^{2^\star - 1}}{\lambda B_{1,\varepsilon} + B_{2,\varepsilon}}.$$

We assume in the sequel that $n \geq 6$.

First, we fix $x_{1,\varepsilon} = x_0$ and we let $\beta_{1,\varepsilon} = \beta_{2,\varepsilon} = \beta_\varepsilon$ verifying that $\beta_\varepsilon \to 1$ as $\varepsilon \to 0$. We choose $x_{2,\varepsilon} \in S^n$ such that $d_h\left(x_{2,\varepsilon}, x_0\right) \geq \left(\beta_\varepsilon - 1\right)^{\frac{1}{20}}$ and such that $d_h\left(x_0, x_{1,\varepsilon}\right) \to 0$ as $\varepsilon \to 0$. One can check by direct computations that h_ε converges to $C(n)S_h$ in $C^0(M)$, with the additional property that the convergence of h_ε is C^1 for $n = 6$. This provides examples of blowing-up sequences of solutions u_ε of Equation (E_ε) for which there is one geometric concentration point carrying two minimal energies. Namely, the concentration points are not isolated.

Second, we choose $x_{1,\varepsilon} = x_{2,\varepsilon} = x_0$ in S^n, $\beta_{1,\varepsilon} = \beta_1$ and we let $\beta_{2,\varepsilon} \to 1$ as $\varepsilon \to 0$. We let also $\lambda = 1$. One checks then by direct computations that the sequence (h_ε) is bounded in $L^\infty(S^n)$ and that $h_\varepsilon \to C(n)S_h$ in $L^p(M)$ for all $p > 1$. Moreover, we clearly have that $\lim_{\varepsilon \to 0} u_\varepsilon = u_0$ weakly in $H_1^2(S_n)$ where $u_0 \not\equiv 0$. This provides examples of blowing-up sequences of solutions u_ε of Equation (E_ε) which does not converge weakly to 0 in dimensions $n \geq 6$.

At last, we choose $x_{1,\varepsilon} = x_{2,\varepsilon} = x_0$ in S^n, $\beta_{1,\varepsilon} = \beta_1$ and we let $\beta_{2,\varepsilon} \to 1$ as $\varepsilon \to 0$. We let also $\lambda > 1$. One checks then by direct computations that the sequence (h_ε) is bounded in $L^\infty(S^n)$ and that $h_\varepsilon \to h_0$ in $L^p(M)$ for all $p > 1$ for some $h_0 \in C^0(M)$. Moreover, when $n \geq 7$, $\inf_{S^n} |h_\varepsilon - C(n)S_h| \to 0$ as $\varepsilon \to 0$ but when $n = 6$, $\inf_{S^n} |h_\varepsilon - C(n)S_h| \not\to 0$ as $\varepsilon \to 0$. Independently, we clearly have that $\lim_{\varepsilon\to 0} u_\varepsilon = u_0$ weakly in $H_1^2(S_n)$ where $u_0 \not\equiv 0$. This provides examples of blowing-up sequences of solutions u_ε of Equation (E_ε) which does not converge weakly to 0 and for which there exists $\varepsilon_0 > 0$ such that $h_\varepsilon > C(n)S_h + \varepsilon_0$ for all $\varepsilon > 0$ in dimension $n = 6$.

We refer to [12] for other examples of blowing-up sequences of solutions of equations like (E).

Appendix 1: An eigenvalue problem

We study in this appendix an eigenvalue problem we used in a crucial way in Sections 2 and 3. The limiting eigenvalue problem, that is the Euclidean one, was studied by Bianchi-Egnell [3] and was used first by Adimurthi-Pacella-Yadava [1] in the study of blow-up problems in the Euclidean space. The lemma below, or more precisely a variant of it, was used in blow-up problems in Riemannian geometry by Druet-Hebey [10] (see also [17]).

Lemma. *We let $(g_\varepsilon)_{\varepsilon>0}$ be a sequence of Riemannian metrics in $B_0(3\delta) \subset \mathbb{R}^n$, $\delta > 0$, verifying that there exists $\lambda > 1$ such that*

$$\lambda^{-1}\xi \leq g_\varepsilon \leq \lambda\xi \tag{A1}$$

in the sense of bilinear forms with ξ the Euclidean metric, that

$$g_\varepsilon(0)_{ij} = \delta_{ij} \tag{A2}$$

for all $i, j \in \{1, \dots, n\}$ and that

$$(g_\varepsilon) \text{ is bounded in } C^2\big(B_0(2\delta)\big)\,. \tag{A3}$$

We let also $(\mu_\varepsilon)_{\varepsilon>0}$ be a sequence of positive real numbers converging to 0 as $\varepsilon \to 0$ and $(\varphi_\varepsilon)_{\varepsilon>0}$ be a sequence of smooth functions with compact support in $B_0(\delta)$ verifying that

$$\mu_\varepsilon^{\frac{n}{2}-1}\varphi_\varepsilon(\mu_\varepsilon x) \to u \text{ as } \varepsilon \to 0 \tag{A4}$$

strongly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ where

$$u(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{1-\frac{n}{2}}.$$

We consider $(\tau_{i,\varepsilon}, \zeta_{i,\varepsilon})$ *the solutions of the following eigenvalue problem:*

$$\text{(A5)} \qquad \begin{cases} \Delta_{g_\varepsilon} \zeta_{i,\varepsilon} = \tau_{i,\varepsilon} \varphi_\varepsilon^{2^\star - 2} \zeta_{i,\varepsilon} \ \text{ in } B_0(\delta) \\ \zeta_{i,\varepsilon} = 0 \ \text{ on } \partial B_0(\delta) \\ \int_{B_0(\delta)} \varphi_\varepsilon^{2^\star - 2} \zeta_{i,\varepsilon} \zeta_{j,\varepsilon} \, dv_{g_\varepsilon} = K_n^{-\frac{n}{2}} \delta_{ij} \end{cases}$$

with $\tau_{1,\varepsilon} \le \dots \le \tau_{i,\varepsilon} \le \dots$ *Then, after passing to a subsequence, we have that*

$$\lim_{\varepsilon \to 0} \tau_{i,\varepsilon} = \tau_i \ \text{ as } \varepsilon \to 0$$

and that

$$\lim_{\varepsilon \to 0} \mu_\varepsilon^{\frac{n}{2} - 1} \zeta_{i,\varepsilon}(\mu_\varepsilon x) \to \zeta_i \ \text{ as } \varepsilon \to 0$$

strongly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ *for all* $i \ge 1$ *where* (ζ_i, τ_i) *satisfy*

$$\text{(A6)} \qquad \begin{cases} \Delta_\xi \zeta_i = \tau_i u^{2^\star - 2} \zeta_i \ \text{ in } \mathbb{R}^n \\ \int_{\mathbb{R}^n} u^{2^\star - 2} \zeta_i \zeta_j \, dv_\xi = K_n^{-\frac{n}{2}} \delta_{ij} \end{cases}$$

and $\tau_1 \le \dots \le \tau_i \le \dots$.

Proof. We prove the result by induction on i. We mainly follow [10] (see also [17]) where a similar result was used. When $i = 1$,

$$\tau_{1,\varepsilon} = \inf_{\varphi \in C_c^\infty(B_0(\delta)),\, \varphi \not\equiv 0} \frac{\int_{B_0(\delta)} |\nabla \varphi|_{g_\varepsilon}^2 \, dv_{g_\varepsilon}}{\int_{B_0(\delta)} \varphi_\varepsilon^{2^\star - 2} \varphi^2 \, dv_{g_\varepsilon}}.$$

Taking $\varphi = \varphi_\varepsilon$, we get by (A1)–(A4) that

$$\limsup_{\varepsilon \to 0} \tau_{1,\varepsilon} \le \frac{\int_{\mathbb{R}^n} |\nabla u|_\xi^2 \, dx}{\int_{\mathbb{R}^n} u^{2^\star} \, dx} = 1.$$

Thus, up to a subsequence, we have that $\lim_{\varepsilon \to 0} \tau_{1,\varepsilon} = \hat{\tau}_1 \le 1$. Thanks to (A1) and (A5), we then get that $\hat{\zeta}_{1,\varepsilon} = \mu_\varepsilon^{\frac{n}{2} - 1} \zeta_{1,\varepsilon}(\mu_\varepsilon x)$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Thus, up to a subsequence, we know that $\hat{\zeta}_{1,\varepsilon} \rightharpoonup \hat{\zeta}_1$ as $\varepsilon \to 0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Thanks to (A2)–(A5), it is easily checked that

$$\Delta_\xi \hat{\zeta}_1 = \hat{\tau}_1 u^{2^\star - 2} \hat{\zeta}_1$$

in $\mathbb{R}^n$ and that

$$\int_{\mathbb{R}^n} u^{2^\star-2}\hat{\zeta}_1^2\,dx = K_n^{-\frac{n}{2}}\,. \tag{A7}$$

It comes from the Euclidean Sobolev inequality and the equation verified by $\hat{\zeta}_1$ that

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\hat{\zeta}_1^{2^\star}\,dx\right)^{\frac{2}{2^\star}} &\le K_n\int_{\mathbb{R}^n}\left|\nabla\hat{\zeta}_1\right|_\xi^2\,dx\\ &= K_n\hat{\tau}_1\int_{\mathbb{R}^n}u^{2^\star-2}\hat{\zeta}_1^2\,dx\\ &\le K_n\hat{\tau}_1\left(\int_{\mathbb{R}^n}u^{2^\star}\,dx\right)^{\frac{2^\star-2}{2^\star}}\left(\int_{\mathbb{R}^n}\hat{\zeta}_1^{2^\star}\,dx\right)^{\frac{2}{2^\star}}\\ &= K_n\hat{\tau}_1\left(\int_{\mathbb{R}^n}\hat{\zeta}_1^{2^\star}\,dx\right)^{\frac{2}{2^\star}}\,.\end{aligned}$$

Since $\hat{\tau}_1\le 1$ and $\hat{\zeta}_1\not\equiv 0$ thanks to (A7), we thus get that $\hat{\tau}_1=1$ and that all the above inequalities are equalities: this implies that $\hat{\zeta}_1=u$. Since, by Bianchi-Egnell [3], $\zeta_1=u$ and $\tau_1=1$, the lemma is proved for $i=1$.

Let $p\ge 2$ and assume that the lemma holds for all $1\le i\le p-1$. We write that

$$\tau_{p,\varepsilon}=\inf_{\varphi\in\mathcal{H}_{p,\varepsilon}}\frac{\int_{B_0(\delta)}|\nabla\varphi|_{g_\varepsilon}^2\,dv_{g_\varepsilon}}{\int_{B_0(\delta)}\varphi_\varepsilon^{2^\star-2}\varphi^2\,dv_{g_\varepsilon}}$$

where

$$\mathcal{H}_{p,\varepsilon}=\left\{\varphi\in C_c^\infty\left(B_0\left(\delta\right)\right)\ \text{ s.t. }\int_{B_0(\delta)}\varphi_\varepsilon^{2^\star-2}\zeta_{i,\varepsilon}\varphi\,dv_{g_\varepsilon}=0\text{ for all }i=1,\dots,p-1\right\}.$$

We claim first that

$$\limsup_{\varepsilon\to 0}\tau_{p,\varepsilon}\le\tau_p. \tag{A8}$$

Let us prove this claim. First, note that we have

$$\tau_p=\inf_{\varphi\in\mathcal{H}_p}\frac{\int_{\mathbb{R}^n}|\nabla\varphi|_\xi^2\,dx}{\int_{\mathbb{R}^n}u^{2^\star-2}\varphi^2\,dx}$$

where

$$\mathcal{H}_p = \left\{\varphi \in C_c^\infty\left(\mathbb{R}^n\right) \text{ s.t. } \int_{\mathbb{R}^n} u^{2^\star-2}\zeta_i\varphi\, dx = 0 \text{ for all } i = 1,\dots,p-1\right\}.$$

Let $f \in \mathcal{H}_p$. We set

$$f_\varepsilon\left(x\right) = \mu_\varepsilon^{1-\frac{n}{2}} f\left(\frac{x}{\mu_\varepsilon}\right)$$

and

$$\hat{f}_\varepsilon\left(x\right) = f_\varepsilon - K_n^{\frac{n}{2}} \sum_{i=1}^{p-1}\left(\int_{B_0(\delta)} \varphi_\varepsilon^{2^\star-2}\zeta_{i,\varepsilon} f_\varepsilon\, dv_{g_\varepsilon}\right)\zeta_{i,\varepsilon}.$$

By (A5), it is clear that, for $\varepsilon > 0$ small enough, $\hat{f}_\varepsilon \in \mathcal{H}_{p,\varepsilon}$. It is easily checked thanks to (A5) that

$$\int_{B_0(\delta)} \varphi_\varepsilon^{2^\star-2}\hat{f}_\varepsilon^2\, dv_{g_\varepsilon} = \int_{B_0(\delta)} \varphi_\varepsilon^{2^\star-2} f_\varepsilon^2\, dv_{g_\varepsilon} - K_n^{\frac{n}{2}}\sum_{i=1}^{p-1}\left(\int_{B_0(\delta)} \varphi_\varepsilon^{2^\star-2}\zeta_{i,\varepsilon} f_\varepsilon\, dv_{g_\varepsilon}\right)^2$$

and that

$$\int_{B_0(\delta)} \left|\nabla \hat{f}_\varepsilon\right|_{g_\varepsilon}^2\, dv_{g_\varepsilon} = \int_{B_0(\delta)} \left|\nabla f_\varepsilon\right|_{g_\varepsilon}^2\, dv_{g_\varepsilon} - K_n^{\frac{n}{2}}\sum_{i=1}^{p-1}\tau_{i,\varepsilon}\left(\int_{B_0(\delta)} \varphi_\varepsilon^{2^\star-2}\zeta_{i,\varepsilon} f_\varepsilon\, dv_{g_\varepsilon}\right)^2.$$

Thanks to (A2), (A3) and to the fact that the lemma holds for $i = 1,\dots,p-1$, we easily get that

$$\int_{B_0(\delta)} \varphi_\varepsilon^{2^\star-2}\zeta_{i,\varepsilon} f_\varepsilon\, dv_{g_\varepsilon} \to 0$$

as $\varepsilon \to 0$ for all $i = 1,\dots,p-1$ and then thanks to (A4) that

$$\int_{B_0(\delta)} \varphi_\varepsilon^{2^\star-2}\hat{f}_\varepsilon^2\, dv_{g_\varepsilon} \to \int_{\mathbb{R}^n} u^{2^\star-2} f^2\, dx$$

as $\varepsilon \to 0$ and

$$\int_{B_0(\delta)} \left|\nabla \hat{f}_\varepsilon\right|_{g_\varepsilon}^2\, dv_{g_\varepsilon} \to \int_{\mathbb{R}^n} \left|\nabla f\right|_\xi^2\, dx.$$

Since it holds for all $f \in \mathcal{H}$, this clearly proves (A8). In particular, $(\tau_{p,\varepsilon})$ is bounded and, up to a subsequence, we have that $\lim_{\varepsilon\to 0}\tau_{p,\varepsilon} = \hat{\tau}_p \le \tau_p$. Thanks to (A1) and (A5), we then get that $\hat{\zeta}_{p,\varepsilon} = \mu_\varepsilon^{\frac{n}{2}-1}\zeta_{p,\varepsilon}(\mu_\varepsilon x)$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Thus, up to a subsequence, we know that $\hat{\zeta}_{p,\varepsilon} \rightharpoonup \hat{\zeta}_p$ as $\varepsilon \to 0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Thanks to (A2)–(A5), it is easily checked that

$$\Delta_\xi \hat{\zeta}_p = \hat{\tau}_p u^{2^\star-2}\hat{\zeta}_p$$

in $\mathbb{R}^n$ and that

$$\int_{\mathbb{R}^n} u^{2^\star-2}\zeta_i\hat{\zeta}_p\,dx = K_n^{-\frac{n}{2}}\delta_{ip}$$

for all $i = 1,\dots,p$. Since $\tau_{p,\varepsilon} \ge \tau_{p-1,\varepsilon}$, we also know that $\hat{\tau}_p \ge \tau_{p-1}$. This clearly implies that $\hat{\tau}_p = \tau_p$ and that $\hat{\zeta}_p = \zeta_p$. This ends the proof of the lemma. q.e.d.

Appendix 2: A Pohozaev identity

We prove in this appendix the Pohozaev identity we repeatedly used in this paper. We let (M,g) be a complete Riemannian manifold and Ω be a compact subset of M with smooth boundary. We let $x_0 \in M$ and $R > 0$ be such that $\Omega \subset B_{x_0}(R)$ and we assume that u is a smooth positive function verifying that

$$\Delta_g u + hu = u^{2^\star-1}$$

in $B_{x_0}(R)$ for some $h \in C^\infty\big(B_{x_0}(R)\big)$. At last, we let $f \in C^\infty\big(B_{x_0}(R)\big)$. Integrating by parts, we have that

$$\begin{aligned}\int_\Omega (\nabla u,\nabla f)_g\,\Delta_g u\,dv_g &= \int_\Omega \Big(\nabla\Big((\nabla u,\nabla f)_g\Big),\nabla u\Big)_g\,dv_g \\ &\quad - \int_{\partial\Omega} (\nabla u,\nabla f)_g\,(\nabla u,\nu)_g\,d\sigma_g\end{aligned}$$

where ν denotes the unit outer normal of $\partial\Omega$ and $d\sigma_g$ is the induced Riemannian measure on $\partial\Omega$. Noting that

$$\Big(\nabla\Big((\nabla u,\nabla f)_g\Big),\nabla u\Big)_g = \nabla^2 f(\nabla u,\nabla u) + \frac{1}{2}\big(\nabla f,\nabla\big(|\nabla u|_g^2\big)\big)_g,$$

we obtain by integration by parts that

$$\begin{aligned}\int_\Omega (\nabla u, \nabla f)_g \,\Delta_g u \,dv_g = \frac{1}{2}\int_\Omega \Delta_g f \,|\nabla u|_g^2 \,dv_g + \int_\Omega \nabla^2 f\big(\nabla u, \nabla u\big)\,dv_g \\ + \frac{1}{2}\int_{\partial\Omega} (\nabla f, \nu)_g \,|\nabla u|_g^2 \,d\sigma_g \\ - \int_{\partial\Omega} (\nabla u, \nabla f)_g \,(\nabla u, \nu)_g \;d\sigma_g\end{aligned}$$

so that

$$\begin{aligned}&\int_\Omega (\nabla u, \nabla f)_g \,\Delta_g u \,dv_g + \frac{n-2}{2}\int_\Omega |\nabla u|_g^2 \,dv_g \\ &= \frac{1}{2}\int_{\partial\Omega} (\nabla f, \nu)_g \,|\nabla u|_g^2 \,d\sigma_g - \int_{\partial\Omega} (\nabla u, \nabla f)_g \,(\nabla u, \nu)_g \;d\sigma_g \\ &\quad + \frac{1}{2}\int_\Omega (\Delta_g f + n)\,|\nabla u|_g^2 \,dv_g + \int_\Omega \big(\nabla^2 f - g\big)\big(\nabla u, \nabla u\big)\,dv_g .\end{aligned}$$

Now we use the equation satisfied by u to get that

$$\int_\Omega |\nabla u|_g^2 \,dv_g = \int_{\partial\Omega} u\,(\nabla u, \nu)_g \,d\sigma_g + \int_\Omega u^{2^\star}\,dv_g - \int_\Omega h u^2 \,dv_g$$

and that

$$\begin{aligned}&\int_\Omega (\nabla u, \nabla f)_g \,\Delta_g u \,dv_g \\ &= \int_\Omega \Delta_g f \left(\frac{1}{2^\star} u^{2^\star} - \frac{1}{2} h u^2\right) dv_g + \frac{1}{2}\int_\Omega (\nabla f, \nabla h)_g \,u^2 \,dv_g \\ &\quad + \int_{\partial\Omega} (\nabla f, \nu)_g \left(\frac{1}{2^\star} u^{2^\star} - \frac{1}{2} h u^2\right) d\sigma_g\end{aligned}$$

which gives that

$$\begin{aligned}&\int_\Omega (\nabla u, \nabla f)_g \,\Delta_g u \,dv_g + \frac{n-2}{2}\int_\Omega |\nabla u|_g^2 \,dv_g \\ &= \int_{\partial\Omega} (\nabla f, \nu)_g \left(\frac{1}{2^\star} u^{2^\star} - \frac{1}{2} h u^2\right) d\sigma_g + \frac{n-2}{2}\int_{\partial\Omega} (\nabla u, \nu)_g \,u\,d\sigma_g \\ &\quad + \int_\Omega \big(\Delta_g f + n\big)\left(\frac{1}{2^\star} u^{2^\star} - \frac{1}{2} h u^2\right) dv_g \\ &\quad + \int_\Omega \left(h + \frac{1}{2}\,(\nabla h, \nabla f)_g\right) u^2 \,dv_g .\end{aligned}$$

Thus we have obtained that

$$\begin{aligned}
&\int_B \left(h + \frac{1}{2}\left(\nabla h, \nabla f\right)_g\right) u^2\, dv_g \\
&= \int_\Omega (\Delta_g f + n) \left(\frac{1}{2}|\nabla u|_g^2 + \frac{1}{2} h u^2 - \frac{1}{2^\star} u^{2^\star}\right)\, dv_g \\
&\quad + \int_\Omega \left(\nabla^2 f - g\right)\left(\nabla u, \nabla u\right) dv_g - \int_{\partial\Omega} \left(\nabla f, \nu\right)_g \left(\frac{1}{2^\star} u^{2^\star} - \frac{1}{2} h u^2\right)\, d\sigma_g \\
&\quad + \frac{1}{2}\int_{\partial\Omega} \left(\nabla f, \nu\right)_g |\nabla u|_g^2\, d\sigma_g - \int_{\partial\Omega} \left(\nabla u, \nabla f\right)_g \left(\nabla u, \nu\right)_g\, d\sigma_g \\
&\quad - \frac{n-2}{2}\int_{\partial\Omega} u\left(\nabla u, \nu\right)_g d\sigma_g.
\end{aligned}$$

Integrating by parts and using the equation satisfied by u, we have that

$$\begin{aligned}
&\int_\Omega (\Delta_g f + n)|\nabla u|_g^2\, dv_g \\
&= \int_\Omega \Big(\nabla\big((\Delta_g f + n)\, u\big), \nabla u\Big)_g dv_g - \frac{1}{2}\int_\Omega \left(\nabla\left(\Delta_g f\right), \nabla u^2\right)_g dv_g \\
&= \int_{\partial\Omega} \left(\Delta_g f + n\right)\left(\nabla u, \nu\right)_g u\, d\sigma_g - \frac{1}{2}\int_{\partial\Omega} \left(\nabla\left(\Delta_g f\right), \nu\right)_g u^2\, d\sigma_g \\
&\quad + \int_\Omega \left(\Delta_g f + n\right)\left(u^{2^\star} - h u^2\right)\, dv_g - \frac{1}{2}\int_\Omega \left(\Delta_g^2 f\right) u^2\, dv_g.
\end{aligned}$$

Thus we get that

$$\begin{aligned}
&\int_\Omega (\Delta_g f + n) \left(\frac{1}{2}|\nabla u|_g^2 + \frac{1}{2} h u^2 - \frac{1}{2^\star} u^{2^\star}\right)\, dv_g \\
&= \frac{1}{n}\int_\Omega (\Delta_g f + n) u^{2^\star}\, dv_g - \frac{1}{4}\int_\Omega \left(\Delta_g^2 f\right) u^2\, dv_g \\
&\quad + \frac{1}{2}\int_{\partial\Omega} \left(\Delta_g f + n\right)\left(\nabla u, \nu\right)_g u\, d\sigma_g - \frac{1}{4}\int_{\partial\Omega} \left(\nabla\left(\Delta_g f\right), \nu\right)_g u^2\, d\sigma_g.
\end{aligned}$$

This finally leads to the following:

$$\begin{aligned}
&\int_\Omega \left(h + \frac{1}{2}\left(\nabla f, \nabla h\right)_g + \frac{1}{4}\left(\Delta_g^2\right) f\right) u^2\, dv_g \\
&= \frac{1}{n}\int_\Omega (\Delta_g f + n) u^{2^\star}\, dv_g + \int_\Omega \left(\nabla^2 f - g\right)\left(\nabla u, \nabla u\right) dv_g + A
\end{aligned}$$

where A is the boundary term

$$\begin{aligned} A = {} & \frac{1}{2}\int_{\partial\Omega} (\nabla f, \nu)_g \, |\nabla u|_g^2 \, d\sigma_g - \int_{\partial\Omega} (\nabla u, \nabla f)_g \, (\nabla u, \nu)_g \, d\sigma_g \\ & - \frac{n-2}{2}\int_{\partial\Omega} (\nabla u, \nu)_g \, u \, d\sigma_g - \int_{\partial\Omega} (\nabla f, \nu)_g \left(\frac{1}{2^\star} u^{2^\star} - \frac{1}{2} h u^2 \right) d\sigma_g \\ & + \frac{1}{2}\int_{\partial\Omega} (\Delta_g f + n) \, (\nabla u, \nu)_g \, u \, d\sigma_g - \frac{1}{4}\int_{\partial\Omega} \left(\nabla\left(\Delta_g f\right), \nu\right)_g u^2 \, d\sigma_g . \end{aligned}$$

This is the relation we referred to as the Pohozaev identity, with test function f, applied in Ω to a function u which verifies the above equation.

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